

Complex form of Fourier Series.

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right]$$

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx,$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi}{L} x dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi}{L} x dx.$$

$$\left. \begin{aligned} e^{i\theta} &= \cos \theta + i \sin \theta \\ e^{-i\theta} &= \cos \theta - i \sin \theta \end{aligned} \right\} \Rightarrow \begin{aligned} \cos \theta &= (e^{i\theta} + e^{-i\theta})/2 \\ \sin \theta &= (e^{i\theta} - e^{-i\theta})/2i \end{aligned}$$

$$\theta_n = \frac{n\pi}{L}$$

$$\Rightarrow f(x) \sim a_0 + \sum_{n=1}^{\infty} \left\{ a_n \left[\frac{e^{i\theta_n x} + e^{-i\theta_n x}}{2} \right] + b_n \left[\frac{e^{i\theta_n x} - e^{-i\theta_n x}}{2i} \right] \right\}$$

$$= a_0 + \frac{1}{2} \sum_{n=1}^{\infty} \left[(a_n - ib_n) e^{i\theta_n x} + (a_n + ib_n) e^{-i\theta_n x} \right]$$

$$= a_0 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n - ib_n) e^{i\theta_n x} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n + ib_n) e^{-i\theta_n x} =$$

$$= a_0 + \frac{1}{2} \sum_{m=-1}^{-\infty} (a_{-m} - ib_{-m}) e^{i\theta_m x} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n + ib_n) e^{-i\theta_n x}$$

$$a_{-m} = a_m^{\text{even}}, \quad b_{-m} = -b_m^{\text{odd}}, \quad \theta_{-m} = -\theta_m$$

$$= a_0 + \frac{1}{2} \sum_{\substack{m=-\infty \\ \downarrow n}}^{-1} (a_m + ib_m) e^{-i\theta_m x} + \sum_{m=1}^{\infty} (a_m + ib_m) e^{-i\theta_m x}$$

$$\Rightarrow f(x) \sim \sum_{n=-\infty}^{\infty} C_n e^{-i\theta_n x}, \quad \begin{aligned} C_0 &= a_0 \\ C_n &= \frac{a_n + ib_n}{2}, \quad n \neq 0 \end{aligned}$$

Similarly, if we change to keep: $e^{i\theta_n x}$.

$$f(x) \sim a_0 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n - ib_n) e^{i\theta_n x} + \frac{1}{2} \sum_{m=-1}^{-\infty} \underbrace{(a_m + ib_{-m})}_{a_m - ib_m} e^{+i\theta_m x}$$

$$\Rightarrow f(x) \sim \sum_{n=-\infty}^{\infty} \bar{C}_n e^{i\theta_n x}, \quad \begin{aligned} \bar{C}_0 &= a_0 \\ \bar{C}_n &= \frac{a_n - ib_n}{2}, \quad n \neq 0 \end{aligned}$$

If $L = \pi$,

$$\Rightarrow f(x) \sim \sum_{n=-\infty}^{\infty} C_n e^{-inx} \quad \text{or} \quad f(x) \sim \sum_{n=-\infty}^{\infty} \bar{C}_n e^{inx}$$

Determination of the coefficients C_n and $\bar{C}_n$ in terms of the function $f(x)$.

(I) If $n \neq 0$
 $n = \dots, -2, -1, 1, 2, \dots$ $C_n = \frac{a_n + ib_n}{2} =$

$$= \frac{1}{2L} \left[\int_{-L}^L f(x) \cos \frac{n\pi}{L} x dx + i \int_{-L}^L f(x) \sin \left(\frac{n\pi}{L} x \right) dx \right] =$$

$$= \frac{1}{2L} \int_{-L}^L f(x) e^{i \frac{n\pi}{L} x} dx$$

for $n=0$ $C_0 = a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx = \frac{1}{2L} \int_{-L}^L f(x) e^{i0x} dx$

(II) If $n \neq 0$ $\bar{C}_n = \frac{a_n - ib_n}{2} =$

$$= \frac{1}{2L} \left[\int_{-L}^L f(x) \cos \frac{n\pi}{L} x dx - i \int_{-L}^L f(x) \sin \frac{n\pi}{L} x dx \right]$$

$$= \frac{1}{2L} \int_{-L}^L f(x) e^{-i \frac{n\pi}{L} x} dx.$$

Summarizing: Complex representation of Fourier Series.

(I)
$$f(x) \sim \sum_{n=-\infty}^{\infty} C_n e^{-i \frac{n\pi}{L} x} \quad (3.1)$$

where
$$C_n = \frac{1}{2L} \int_{-L}^L f(x) e^{i \frac{n\pi}{L} x} dx. \quad (3.2)$$

or

(II)
$$f(x) \sim \sum_{n=-\infty}^{\infty} C_n e^{i \frac{n\pi}{L} x} \quad (3.3)$$

where
$$C_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-i \frac{n\pi}{L} x} dx \quad (3.4)$$

Transition from a discrete set of eigenvalues to a continuous set.

Substitution of (3.2) into (3.1) leads to

$$f(x) \sim \sum_{n=-\infty}^{\infty} C_n e^{-i \frac{n\pi}{L} x} \Rightarrow$$

$$\boxed{f(x) \sim \sum_{n=-\infty}^{\infty} \left[\frac{1}{2L} \int_{-L}^L f(\bar{x}) e^{i \frac{n\pi}{L} \bar{x}} d\bar{x} \right] e^{-i \frac{n\pi}{L} x}} \quad (3.5)$$

Fourier series identity.

Calling $\omega = \frac{n\pi}{L}$ and $\Delta\omega = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$
 $n = \dots -2, -1, 0, 1, 2, \dots \Rightarrow \frac{1}{2L} = \frac{\Delta\omega}{2\pi}$

Subst. into (3.5)

$$f(x) \sim \sum_{n=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \left[\int_{-L}^L f(\bar{x}) e^{i\omega\bar{x}} d\bar{x} \right] e^{-i\omega x} \quad (3.6)$$

when $L \rightarrow \infty$, the distance between consecutive, $\omega = \sqrt{\lambda}$,
 $\Delta\omega \rightarrow 0$ $\Rightarrow$ that all real numbers are possible ω .

Therefore, when $L \rightarrow \infty$ equ. (3.6) can be written

as

$$f(x) \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\bar{x}) e^{i\omega \bar{x}} d\bar{x} \right] e^{-i\omega x} d\omega \quad (3.7)$$

"Fourier integral identity."

We claim that the two ^{improper} integrals in the (3.7) expression converge if $f(x)$ is absolutely integrable: $\int_{-\infty}^{\infty} |f(x)| dx < \infty$

For example, if $f(x)$ is piecewise smooth and $f(x) \rightarrow 0$ as $x \rightarrow \pm\infty$

sufficiently fast, then $f(x)$ is absolutely integrable.

Now, we are ready for our definition:

Def - The function

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x}) e^{i\omega \bar{x}} d\bar{x}$$

is called the Fourier Transform of $f(x)$.

and equ. (3.7) can be written as

$$f(x) \sim \int_{-\infty}^{\infty} F(\omega) e^{-i\omega x} d\omega$$

Inverse Fourier
transform of $F(\omega)$

Comparison

For $f(x)$ piecewise smooth on $[-L, L]$.

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{-\frac{i n \pi}{L} x}$$

$$C_n = \frac{1}{2L} \int_{-L}^L f(x) e^{\frac{i n \pi}{L} x} dx.$$

Discrete Fourier Series of $f(x)$ on $[-L, L]$.

or

$$f(x) = \sum_{n=-\infty}^{\infty} \bar{C}_n e^{\frac{i n \pi}{L} x}$$

$$\bar{C}_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-\frac{i n \pi}{L} x} dx$$

For $f(x)$ absolutely integrable on $(-\infty, \infty)$.

$$f(x) = \int_{-\infty}^{\infty} C(\omega) e^{-i \omega x} d\omega$$

Inverse Fourier transform.

$$C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x}) e^{i \omega \bar{x}} d\bar{x}$$

Fourier transform of $f(x)$

$$f(x) = \int_{-\infty}^{\infty} C(\omega) e^{i \omega x} d\omega$$

$$C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x}) e^{-i \omega \bar{x}} d\bar{x}$$

Application to Heat Equation on an infinite domain.

$$\begin{cases} u_t = K u_{xx}, & -\infty < x < \infty, \quad t > 0 & (5.0) \\ u(x, 0) = f(x) & & (5.0.1) \\ + \text{B.C.'s at } \infty : u(-\infty, t) = 0, \quad u(+\infty, t) = 0 & & (5.0.2) \end{cases}$$

Separation of variables leads to

$$u(x, t) = \phi(x) h(t)$$

$$(I) \quad \frac{dh}{dt} + \lambda K h = 0.$$

$$(II) \quad \text{EVP: } \begin{cases} \frac{d^2 \phi}{dx^2} + \lambda \phi = 0 & (5.1) \\ |\phi(-\infty)| < \infty, \quad |\phi(+\infty)| < \infty & (5.2) \end{cases}$$

Why not $\phi(+\infty) = 0, \quad \phi(-\infty) = 0$?

In order to satisfy (5.1) and (5.2) $\lambda \geq 0$
 (If $\lambda < 0$, $\phi(x)$ would be a combination of exponentials
 and won't be bounded as $x \rightarrow \infty$ and $x \rightarrow -\infty$ at the same time.)

Thus,
$$\phi(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)$$

Satisfies both (5.1) and (5.2) for all $\lambda \geq 0$.

All $\lambda > 0$ are eigenvalues of (5.1) and (5.2)

'Continuous spectrum'.

Now, by extending the superposition principle from the discrete to the continuous case, we obtain

$$u(x,t) = \int_0^{\infty} [C_1(\lambda) \cos(\sqrt{\lambda}x) e^{-\lambda \kappa t} + C_2(\lambda) \sin(\sqrt{\lambda}x) e^{-\lambda \kappa t}] d\lambda$$

using $\lambda = \omega^2$

$$u(x,t) = \int_0^{\infty} [A(\omega) \cos(\omega x) e^{-\kappa \omega^2 t} + B(\omega) \sin(\omega x) e^{-\kappa \omega^2 t}] d\omega$$

or introducing complex functions.

$$u(x,t) = \int_{-\infty}^{\infty} C(\omega) e^{-i\omega x} e^{-\kappa \omega^2 t} d\omega$$

where (6.1)

$$C(\omega) = \begin{cases} \frac{A(-\omega) - iB(-\omega)}{2}, & \omega < 0 \\ \frac{A(\omega) + iB(\omega)}{2}, & \omega > 0 \end{cases}$$

To satisfy I.C.

$$u(x,0) = f(x) = \int_{-\infty}^{\infty} C(\omega) e^{-i\omega x} d\omega$$

(6.2)

See next two pages

If $f(x)$ is absolutely integrable, then (6.2) is the inverse Fourier transform of $f(x)$ and $C(\omega)$ is its Fourier transform

From the previous definition,

$$C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x}) e^{i\omega \bar{x}} d\bar{x} \quad (6.3)$$

Therefore, the solution of the Heat conduction equation on an infinite domain with BC's: $U(\pm\infty, t) = 0$ is given by the pair:

$$\begin{cases} U(x, t) = \int_{-\infty}^{\infty} C(\omega) e^{-i\omega x} e^{-k\omega^2 t} d\omega \end{cases} \quad (7.1)$$

$$\begin{cases} C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x}) e^{i\omega \bar{x}} d\bar{x} \end{cases} \quad (7.2)$$

It's possible to combine these two into a single integral representation:

$$U(x, t) = \int_{-\infty}^{\infty} f(\bar{x}) \frac{1}{\sqrt{4\pi k t}} e^{-(x-\bar{x})^2/4kt} d\bar{x}$$

In fact, substitution of (7.2) into (7.1) leads to

$$U(x, t) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x}) e^{i\omega \bar{x}} d\bar{x} \right] e^{-i\omega x} e^{-k\omega^2 t} d\omega \quad (7.3)$$

interchanging
order of
integration $= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\bar{x}) \left[\int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega(x-\bar{x})} d\omega \right] d\bar{x}$

The expression in brackets

$$\int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega(x-\bar{x})} d\omega = g(x-\bar{x})$$

where $\boxed{g(x) = \int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega x} d\omega}$

is the ^{Fourier} inverse transform of $e^{-k\omega^2 t}$

Inverse Fourier Transform of a Gaussian

Consider $\hat{G}(\omega) = e^{-\alpha\omega^2}$

and its inverse F.T.

$$\hat{g}(x) = \int_{-\infty}^{\infty} e^{-\alpha\omega^2} e^{-i\omega x} d\omega$$

we will prove first that

$$\boxed{\hat{g}'(x) = -\frac{x}{2\alpha} \hat{g}(x)} \quad \text{1st order linear ODE.}$$

In fact,

$$\begin{aligned} \hat{g}'(x) &= \int_{-\infty}^{\infty} -i\omega e^{-\alpha\omega^2} e^{-i\omega x} d\omega = \dots \\ &= \int_{-\infty}^{\infty} \frac{-i\omega}{2\alpha} 2\alpha e^{-\alpha\omega^2} e^{-i\omega x} d\omega = \frac{i}{2\alpha} \int_{-\infty}^{\infty} \frac{d}{d\omega} (e^{-\alpha\omega^2}) e^{-i\omega x} d\omega \end{aligned}$$

Integration by parts

$$\begin{aligned}\hat{g}'(x) &= \frac{i}{2\alpha} \left[e^{-\alpha\omega^2} e^{-i\omega x} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (i\omega) e^{-\alpha\omega^2} e^{-i\omega x} d\omega \\ &= -\frac{x}{2\alpha} \int_{-\infty}^{\infty} e^{-\alpha\omega^2} e^{-i\omega x} d\omega = -\frac{x}{2\alpha} \hat{g}(x) \checkmark\end{aligned}$$

Solving this ODE:

$$\boxed{\hat{g}(x) = C e^{x^2/4\alpha} = \hat{g}(0) e^{x^2/4\alpha}}$$

where $\hat{g}(0) = \int_{-\infty}^{\infty} e^{-\alpha\omega^2} d\omega$

Calling: $z^2 = \alpha\omega^2 \Rightarrow \omega = \frac{z}{\sqrt{\alpha}} \Rightarrow d\omega = \frac{1}{\sqrt{\alpha}} dz$

$$\Rightarrow \boxed{\hat{g}(0) = \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} e^{-z^2} dz =}$$

Since

$$\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy} =$$

$$\begin{aligned}&= \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy \right]^{1/2} \text{ Using polar coords} = \\ &= \left[\int_0^{\infty} \int_0^{2\pi} e^{-r^2} r d\theta dr \right]^{1/2} = \left[2\pi \int_0^{\infty} r e^{-r^2} dr \right]^{1/2} = \left[-\pi \int_0^{\infty} 2r e^{-r^2} dr \right]^{1/2} \\ &= \left[-\pi e^{-r^2} \right]_0^{\infty}^{1/2} = (\pi)^{1/2} \checkmark\end{aligned}$$

Finally,

If $\boxed{\hat{G}(\omega) = e^{-\alpha\omega^2}}$ its inverse F.T. is

given by

$$\boxed{\hat{g}(x) = \sqrt{\frac{\pi}{\alpha}} e^{-x^2/4\alpha}}$$

Therefore,

$$g(x) = \int_{-\infty}^{\infty} e^{-k\omega^2 t} e^{-i\omega x} d\omega = \sqrt{\frac{\pi}{kt}} e^{-\frac{x^2}{4kt}} \quad \alpha = kt$$

Substitution into (7.3)

$$u(x,t) = \frac{1}{2\pi} \frac{\sqrt{\pi}}{\sqrt{kt}} \int_{-\infty}^{\infty} f(\bar{x}) e^{-\frac{(x-\bar{x})^2}{4kt}} d\bar{x}$$

or $\boxed{u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-\bar{x})^2}{4kt}} f(\bar{x}) d\bar{x}.}$