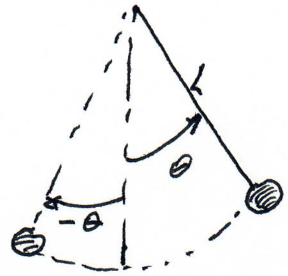
2.16 Nonlinear equations

Pendulum motion:



$$\theta''(t) = -(g/L)\sin(\theta(t))$$

g : gravit. const. L : length of rigid bar

If $\frac{g}{L} \equiv 1$, then

$$\boxed{\theta''(t) = -\sin(\theta(t))} \qquad (1)$$

Typical problems for the pendulum are IVP's.:

$$\begin{cases} \theta''(t) = -\sin(\theta(t)), & t>0 \qquad (1) \\ \theta(0)=\alpha, \quad \theta'(0)=\mu & \qquad (2) \end{cases}$$

But, we can also consider a BVP:

$$\begin{cases} \theta''(t) = -\sin(\theta(t)), & 0<t<T \\ \theta(0)=\alpha, \quad \theta(T)=\beta \end{cases}$$

It means that a pend. that starts at an angle α, from the equill. position $\theta=0$, will be at a position $\theta=\beta$ at a specified time $t=T$.

LINEAR PROBLEM: For small angles, it's possible to approx. $\sin\theta \simeq \theta$

$$\begin{cases} \theta''(t) = -\theta(t), & 0<t<T \qquad (3) \\ \theta(0)=\alpha, \quad \theta(T)=\beta & \qquad (4) \end{cases}$$

LINEAR PROBLEM

For small θ $(\theta \simeq 0)$ $\quad \sin\theta \simeq \theta$

$$\begin{cases} \theta''(t) = \theta(t), & 0 < t < T \\ \theta(0) = \alpha, \quad \theta(T) = \beta \end{cases}$$

General Soln:

$$\theta(t) = A\cos t + B\sin t.$$

Using B.C. at $t=0$

$$\alpha = \theta(0) = A \Rightarrow \boxed{A=\alpha}$$

then $\boxed{\theta(t) = \alpha\cos t + B\sin t}$

CASES:

① $T \neq n\pi, \quad n = 1, 2, \ldots$

$$\beta = \theta(T) = \alpha\cos T + \beta\sin T$$

$$\Rightarrow \quad B = \frac{\beta - \alpha\cos T}{\sin T}$$

$$\Rightarrow \boxed{\theta(t) = \alpha\cos t + \left(\frac{\beta - \alpha\cos T}{\sin T}\right)\sin t.}$$ Unique Solution !

② $T = n\pi$ $(n = 1, 2, \ldots)$ and $\alpha \neq \beta$.

For example $T = 2\pi$ (Similar for other multiples of π).

$$\beta = \theta(T) = \alpha\cos(2\pi) + B\sin(2\pi) = \alpha$$

which is impossible if $\alpha \neq \beta$.

No Solution !

③ $T = 2\pi$ and $\alpha = \beta$

$\beta = \theta(T) = \alpha \cos(2\pi) + B \sin(2\pi) = \alpha$

B.C. at $t = T$ is satisfied because $\alpha = \beta$

but B remains undetermined

then

$\boxed{\theta(t) = \alpha \cos t + B \sin t}$ infinitely many solutions.

Before Maple for Linear Problem.

Consider
$$\begin{cases} \theta''(t) = -\theta(t), & 0 < t < 2\pi \\ \theta(0) = 1, \quad \theta(2\pi) = 1 \end{cases}$$

Infinitely many solns.

$$\theta(t) = \cos t + B \sin t, \qquad B \in \mathbb{R}.$$

Initial angular speed

In particular, Three different Solutions

$$\boxed{\begin{array}{ll} \theta_1(t) = \cos t & \Rightarrow \theta_1'(0) = 0 \\ \theta_2(t) = \cos t + \sin t & \Rightarrow \theta_2'(0) = 1 \\ \theta_3(t) = \cos t - \sin t & \Rightarrow \theta_3'(0) = -1 \end{array}}$$

SHOW MAPLE and establish a relationship with Curious person watching through window every T minutes Joke! physiology of two persons involved.

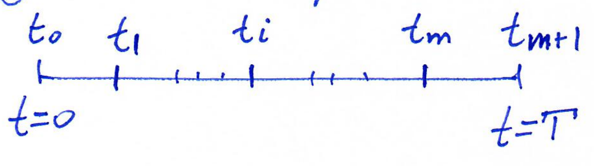
Homework has to do with behavior of numerical solns. when $h \to 0$ and ~~when~~ there are infinitely many solns. As in this case.

It's interesting to observe that the pendulum balls meet at $t = \pi$ for all three experiments. However, they follow different paths after that (more negative angle, or increasing from negative to positive but at different angular speeds and the three of them meet again at $t = 2\pi$.

Consider the nonlinear problem:

$$\begin{cases} \theta''(t) = -\sin(\theta(t)), & 0 < t < T \quad (1) \\ \theta(0) = \alpha, \quad \theta(T) = \beta & (2) \end{cases}$$

Discretization: Grid points



$$h = \frac{T}{m+1}$$

Using centered differences for $\theta''(t)$ (1)-(2) transforms into

$$\boxed{\frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2} + \sin\theta_i = 0, \quad i = 1, 2, .., m}$$

$\underline{\underline{i=1}}$
$$\frac{1}{h^2}\left(-2\theta_1 + \theta_2\right) + \sin\theta_1 + \frac{1}{h^2}\overset{=\alpha}{\theta_0} = 0 \qquad (3)$$

or $\boxed{G_1(\theta_1, \theta_2, \dots, \theta_m) = 0}$

$\underline{\underline{i=2}}$
$$\frac{1}{h^2}\left(\theta_1 - 2\theta_2 + \theta_3\right) + \sin\theta_2 = 0 \qquad (4)$$

or $\boxed{G_2(\theta_1, \theta_2, \theta_3, \dots, \theta_m) = 0}$

$\vdots$

$\underline{\underline{i=m}}$
$$\frac{1}{h^2}\left(-2\theta_m + \theta_{m-1}\right) + \sin\theta_m + \frac{1}{h^2}\overset{=\beta}{\theta_{m+1}} = 0 \qquad (5)$$

or $\boxed{G_m(\theta_1, \dots \theta_m) = 0}$

Defining

$$\vec{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_m \end{bmatrix}, \qquad \vec{G}(\vec{\theta}) = \begin{bmatrix} G_1(\vec{\theta}) \\ G_2(\vec{\theta}) \\ \vdots \\ G_m(\theta) \end{bmatrix}$$

The previous <u>nonlinear system of equations</u> can be represented by

$$\vec{G}(\vec{\theta}) = \vec{0} \qquad (3.1)$$

Therefore, to find a solution of (1)-(2) is equivalent to find a root of (3.1)

For this purpose, we will use Newton method.

For System of equs. $\vec{G}(\vec{\theta}) = \vec{0}$ Find $\vec{\theta}$

Newt. method:

Taylor Exp.: $\vec{G}(\vec{\theta}^{k+1}) = \vec{G}(\vec{\theta}^{k}) + J(\vec{\theta}^{k})(\vec{\theta}^{k+1} - \vec{\theta}^{k}) + \mathcal{O}(\|\vec{\theta}^{k+1} - \vec{\theta}^{k}\|^2)$

If $\vec{\theta}^{k+1}$ is root of $\vec{G}$, then

or $\vec{\theta}^{(k+1)} = \vec{\theta}^{(k)} - J^{-1}(\vec{\theta}^{k})\vec{G}(\vec{\theta}^{(k)})$

$$\boxed{\vec{G}(\vec{\theta}^{k}) + J(\vec{\theta}^{k})(\vec{\theta}^{k+1} - \vec{\theta}^{k}) = 0} \quad \boxed{(5)}$$ dropping higher order Terms

where $J(\vec{\theta}^{k})$ is the Jacobian matrix evaluated at $\vec{\theta}^{k}$.

$$J_{ij}(\vec{\theta}) \equiv \frac{\partial G_i}{\partial \theta_j}(\vec{\theta}), \qquad i = 1, 2, .., m$$

Therefore, for the pendulum BY problem

$$J_{ij}(\vec{\theta}) = \begin{cases} 1/h^2, & j = i-1 \text{ or } j = i+1 \\ -2/h^2 + \cos\theta_i, & j = i \\ 0, & \text{otherwise} \end{cases} \qquad i = 2, .., m-1$$

$$J_{1j}(\vec{\theta}) = \begin{cases} 1/h^2, & j = 2 \\ -2/h^2 + \cos\theta_1, & j = 1 \\ 0, & \text{otherwise} \end{cases} \qquad i = 1$$

$$J_{mj}(\vec{\theta}) = \begin{cases} 1/h^2, & j = m-1 \\ -2/h^2 + \cos\theta_m, & j = m \\ 0, & \text{otherwise} \end{cases} \qquad i = m$$

Recall,

$$\boxed{G_i(\theta_1, \theta_2, .., \theta_m) = \frac{1}{h^2}\left[\theta_{i-1} - 2\theta_i + \theta_{i+1}\right] + \sin\theta_i}$$

$\theta_0 = \alpha$, $\theta_{m+1} = \beta$, $i = 1, 2, .., m$

thus,

$$J(\vec{\theta}) = \frac{1}{h^2}\begin{bmatrix} -2+h^2\cos\theta_1 & 1 & 0 & 0 & 0 \cdots 0 \\ 1 & -2+h^2\cos\theta_2 & 1 & 0 & \cdots 0 \cdots 0 \\ 0 & \ddots & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & \ddots \\ 0 & \cdots & & 1 & -2+h^2\cos\theta_m \end{bmatrix}_{m\times m}$$

From (5) $\boxed{\vec{\theta}^{(k+1)} = \vec{\theta}^{k} - J^{-1}(\vec{\theta}^{k})\,\vec{G}(\vec{\theta}^{k})}$ Not use in practice

Alternative: $\boxed{\vec{\theta}^{(k+1)} = \vec{\theta}^{(k)} + \vec{S}^{(k)}}$ $\longrightarrow$ STEP 2

where $\vec{S}^{k}$ is the Soln. for linear syst. $\longrightarrow$ STEP 1

$$\boxed{J(\vec{\theta}^{k})\,\vec{S}^{(k)} = -\vec{G}(\vec{\theta}^{(k)})}$$

Avoid Comp. of $J^{-1}(\vec{\theta}^{k})$

Application:

$T = 2\pi$, $\alpha = \beta = 0.7$

A) Need init. guess: a) $\vec{\theta}^{(0)} = \begin{pmatrix} 0.7\cos\theta_1 + 0.5\sin\theta_1 \\ \vdots \\ 0.7\cos\theta_m + 0.5\sin\theta_m \end{pmatrix}$ where $\theta_i = i * h$.

b) $\vec{\theta}^{(0)} = \begin{pmatrix} 0.7\cos\theta_1 \\ \vdots \\ 0.7\cos\theta_m \end{pmatrix}$ Fig. 2.4(b)

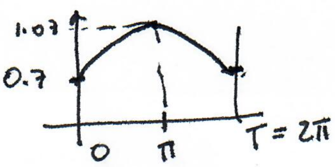
Figure 2.4 (a) in Textbook

Comments on speed of convergence. SHOW BOOK GRAPH 2.4 and Table 2.1

quadratic Converg. Newt.: $\|\vec{\theta}^{k+1} - \vec{\theta}^{k}\| \leq C\|\vec{\theta}^{k} - \vec{\theta}^{k-1}\|^2$

B) OTHER soln. Different guess

$\theta_i^{(0)} \equiv 0.7 + \sin(t_i/2)$ No oscilation



leads to a diff. soln.

SHOW Graph 2.5

Accuracy and Stability

Nonlinear Pend.:

$$\boxed{\begin{cases} \theta''(t) = -\sin(\theta(t)) & (1) \\ \theta(0) = \alpha, \quad \theta(T) = \beta \end{cases}}$$

FD centered:

$$\boxed{G_i(\theta_1, \theta_2, \ldots, \theta_m) = \frac{1}{h^2}\left[\theta_{i-1} - 2\theta_i + \theta_{i+1}\right] + \sin\theta_i = 0 \quad i = 1,\ldots,m} \qquad (2)$$

Def.- (L.T.E.)

If $\hat{\theta}(t)$ were the exact soln. for (1)

by substituting it into (2) we get an extra term

this term is called the L.T.E of the numerical method (2).

For this pendulum problem

$$\boxed{L.T.E = \tau_i(h) = G_i(\hat{\theta}^h)} \iff \boxed{\vec{G}(\hat{\theta}^h) = \vec{\tau}^h} \qquad (3)$$

Now,

$$\vec{G}(\hat{\theta}^h) = \frac{1}{h^2}\left[\hat{\theta}(t_{i-1}) - 2\hat{\theta}(t_i) + \hat{\theta}(t_{i+1})\right] + \sin(\hat{\theta}(t_i)) =$$

done before Taylor exp.

$$= \cancel{\hat{\theta}''(t_i)} + \frac{1}{12}h^2\hat{\theta}''''(t_i) + \cancel{\sin(\hat{\theta}_i)} + O(h^4).$$

$$= \frac{1}{12}h^2\hat{\theta}''''(t_i) + O(h^4).$$

$$\therefore \quad \boxed{\tau_i(h) = \frac{1}{12}h^2\hat{\theta}'''' + O(h^4)}$$

$$\Rightarrow G_i(\hat{\theta}^h) = \frac{1}{12}h^2\hat{\theta}''''(t_i) + \cdots$$

Def.— (Global error)

$$\vec{E}^h \equiv \underbrace{\vec{\theta}^h}_{\text{Numerical soln}} - \underbrace{\hat{\theta}^h}_{\text{Exact solution}} = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_m \end{pmatrix} - \begin{pmatrix} \hat{\theta}_1 \\ \vdots \\ \hat{\theta}_m \end{pmatrix}$$

$$\|\vec{E}^h\| = \|\vec{\theta}^h - \hat{\theta}^h\|$$

Thm.— $\boxed{\vec{E}^h \simeq -J^{-1}(\hat{\theta}^h)\,\vec{\tau}^h}$ if $\underbrace{\|\vec{E}^h\| \text{ is small.}}$

↳ comment on this assumption!

Proof.— Expanding $\vec{G}(\vec{\eta})$ in Taylor polynomial around $\vec{\eta} = \hat{\theta}^h$ ↑exact soln.

$$\vec{G}(\vec{\eta}) = \vec{G}(\hat{\theta}^h) + J(\hat{\theta}^h)(\vec{\eta} - \hat{\theta}^h) + \mathcal{O}(\|\vec{\eta} - \hat{\theta}^h\|^2)$$

In particular, at $\vec{\eta} = \vec{\theta}^h$ ↑numerical soln.

$$0 = \vec{G}(\vec{\theta}^h) = \vec{G}(\hat{\theta}^h) + J(\hat{\theta}^h)(\vec{\theta}^h - \hat{\theta}^h) + \mathcal{O}(\|\vec{\theta}^h - \hat{\theta}^h\|^2)$$

$$\Rightarrow \quad J(\hat{\theta}^h)(\vec{\theta}^h - \hat{\theta}^h) = -\vec{G}(\hat{\theta}^h) + \mathcal{O}(\|\quad\|^2)$$

or $$J(\hat{\theta}^h)\,\vec{E}^h \overset{(3)}{=} -\vec{\tau}^h + \mathcal{O}(\|\vec{E}^h\|^2)$$

$$\Rightarrow \quad \boxed{\vec{E}^h \simeq -J^{-1}(\hat{\theta}^h)\,\vec{\tau}^h} \qquad (4)$$

Def- The nonlinear FD method

$$\vec{G}(\vec{\theta}^h)=\vec{0}$$

is stable in $\|\cdot\|$ if

① There exists $J^{-1}(\hat{\theta}^h)$, for all $h<h_0$.

② There is C such that

$$\|J^{-1}(\hat{\theta}^h)\|\leq C, \quad h<h_0.$$

Thm.- If the nonlinear FDM $\vec{G}(\vec{\theta}^h)=0$ is stable and consistent in $\|\cdot\|$ then, it is convergent

$$\|\vec{E}^h\| \xrightarrow[h\to 0]{} 0.$$

Rmk: Not obvious our previous assumption of dropping the term of order $\mathcal{O}(\|\vec{\theta}^h-\hat{\theta}^h\|^2)$ to obtain a linear system (4) for $\vec{E}^h$!