Newton Method

We want to approx. roots of

$$f(x) = 0$$

Theorem.-

① $f \in C^2[a,b]$

② there exists $p \in [a,b]$ such that $f(p) = 0$.

③ $f'(p) \neq 0$.

Then,

① There exists $\delta > 0$, s.t.
if $p_0 \in [p-\delta, p+\delta] \Rightarrow$

$$x_{n+1} = g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)} \underset{n\to\infty}{\longrightarrow} p \quad \text{(Newt. iteration)}$$

② The sequence $\{x_n\}_1^\infty$ converges to p quadratically.
if $f \in C^3[a,b]$.

Proof. of ②

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$\Rightarrow \quad g'(x) = 1 - \frac{(f'(x))^2 - f(x)\, f''(x)}{(f'(x))^2} = \cancel{1} - \cancel{1} - \frac{f(x) f''(x)}{(f'(x))^2} =$$

Then,

$$g'(p) = \frac{-\overset{=0}{f(p)}\, f''(p)}{\underset{\neq 0}{(f'(p))^2}} = 0$$

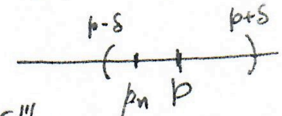
Also, using Taylor thm for $x \in (p-\delta, p+\delta)$

$$g(x) = \underset{=p}{g(p)} + \underset{=0}{g'(p)}(x-p) + g''(\xi(x))\,\frac{(x-p)^2}{2}$$

for $x = p_n$

$$p_{n+1} = g(p_n) = p + g''(\xi(p_n))\,\frac{(p_n-p)^2}{2},$$

$\xi(p_n)$ p_n and p



$$\Rightarrow \quad |p_{n+1} - p| = \frac{g''(\xi(p_n))}{2}\,|p_n - p|^2$$

Cont. of $f'''(x)$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^2} = \frac{1}{2}\lim_{n\to\infty} g''(\xi(p_n)) = \frac{1}{2}g''(p)$$

or $$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^2} = \frac{1}{2}g''(p)$$ or $\boxed{p_n \to p \text{ Quadratically}}$

Example:

a) $p_n = \dfrac{1}{10^{2^n}}$

$$p_n \to p = 0$$

and $\dfrac{|p_{n+1} - p|}{|p_n - p|} = \dfrac{10^{2^n}}{10^{2^{n+1}}} = \dfrac{10^{2^n}}{10^{2^n \cdot 2}} = \dfrac{10^{2^n}}{\left(10^{2^n}\right)^2} = \dfrac{1}{10^{2^n}}$

Also, $\dfrac{|p_{n+1} - p|}{|p_n - p|^2} = \dfrac{\left(10^{2^n}\right)^2}{10^{2^{n+1}}} = \dfrac{10^{2^{n+1}}}{10^{2^{n+1}}} = 1$

$$\Rightarrow \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = 1$$

$\Rightarrow \quad p_n = \dfrac{1}{10^{2^n}} \xrightarrow[n\to\infty]{} 0$ quadratically

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b) $p_n = \dfrac{1}{10^{2^{2n}}}, \quad p_n \to p = 0$

$$\frac{|p_{n+1} - 0|}{|p_n - 0|} = \frac{10^{2^{2n}}}{10^{2^{2n+2}}} = \frac{10^{2^{2n}}}{10^{2^{2n} \cdot 2^2}} = \frac{10^{2^{2n}}}{\left(10^{2^{2n}}\right)^4}$$

$\Rightarrow \quad p_n \to p$ of order 4.

Relative speed of convergence.

n	$p_n = \frac{1}{n}$	$p_n = \frac{1}{n^2}$	$p_n = \frac{1}{10^{2^n}}$
1	1	1	1
2	5×10^{-1}	2.5×10^{-1}	10^{-4}
3	3.3×10^{-1}	1.1×10^{-1}	10^{-8}
4	2.5×10^{-1}	6.25×10^{-2}	10^{-16}
5	2×10^{-1}	4×10^{-2}	9.9×10^{-33}
⋮	⋮	⋮	⋮
9	1.1×10^{-1}	1.23×10^{-2}	$9.\bar{9} \times 10^{-129}$
10	10^{-1}	10^{-2}	10^{-256}

Quadratic Convergence of Newton Method.

Consider the equation

$$\boxed{f(x) = 0,}$$

where $f \in C^3[a,b]$ and there exists $p \in (a,b)$ such that

$$\boxed{f(p) = 0}$$

Also,

$$\boxed{f'(p) \neq 0}$$

Orders of Convergence.

Assume $\{x_n\}_{n=1}^{\infty}$ is converging to x^*.

(I) $\{x_n\}$ converges to x^* at least linear
if there are $0 < c < 1$, $N > 0$ such that

$$|x_{n+1} - x^*| \le c\,|x_n - x^*|, \quad n \geq N.$$

(II) $\{x_n\}$ Converges to x^* at least Superlinear
if there exist a sequence $\varepsilon_n \to 0$ and $N > 0$
Such that

$$|x_{n+1} - x^*| \le \varepsilon_n\,|x_n - x^*|$$

(III) $\{x_n\}$ Converges to x^* at least quadratic
if there are a constant C (not necessarily less than 1) and an integer N such that

$$|x_{n+1} - x^*| \le C\,|x_n - x^*|^2$$