# 1.3 Vector Equations.

## Basic definition of Vectors

Ordered list of numbers

A convenient representation can be obtained from matrices. In fact, a matrix consisting only of one column may be used to represent a Vector. For instance,

$$\vec{u}^{(*)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

They are also called Column vectors.

## Vectors in $\mathbb{R}^2$

Set of Vectors with only two real numbers as entries is denoted by $\mathbb{R}^2$, so

$$\mathbb{R}^2 = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \ u_1, u_2 \text{ are real numbers} \right\}$$

## Addition and scalar multiplication

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{then} \quad \vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

$$c\vec{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix},$$ where $c$ is a real number and $c$ is called a scalar.

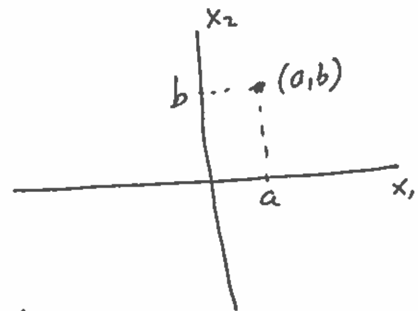
(*) Note: In our textbook they use bold-face letters: $\mathbf{u}, \mathbf{v}, \mathbf{w}$ to represent Vectors. We will use instead $\vec{u}, \vec{v}, \vec{w}$ for the same purpose

Exercise #2

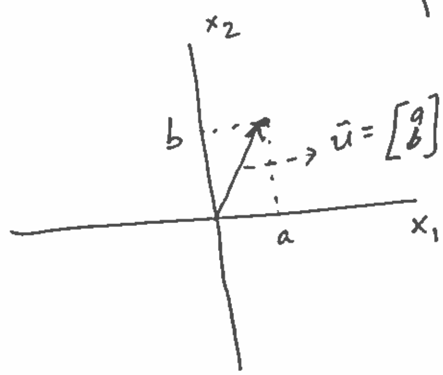
$\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\quad \vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, then $\quad \vec{u} - 2\vec{v} = \begin{bmatrix} 3 - 2(2) \\ 2 - 2(-1) \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

## Geometric Description of $\mathbb{R}^2$

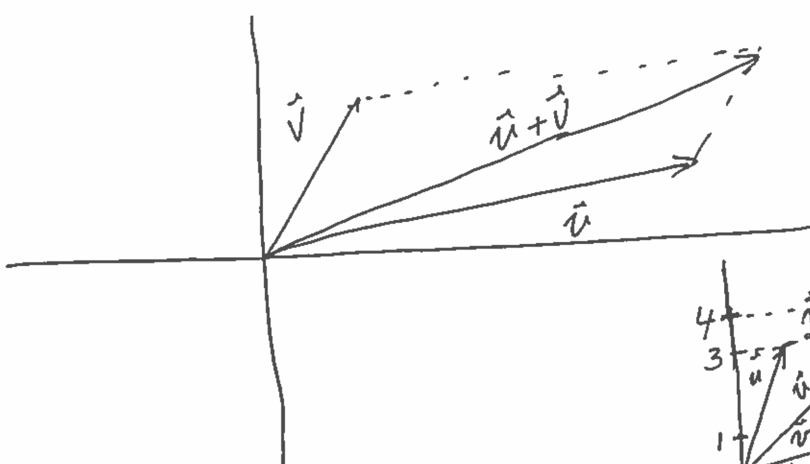
Points in the plane



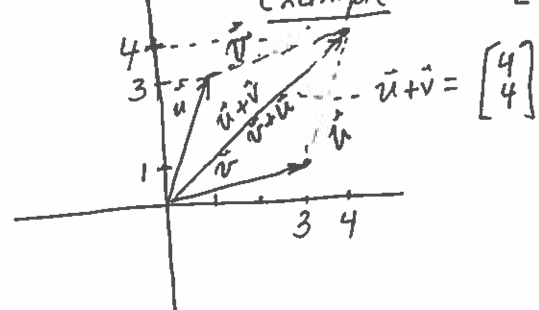

Representation of Vectors in the plane by arrows from the origin of coordinates to the point (a,b)




## Geometric Representation of the Sum of two vectors




Example $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$



# Vectors in $\mathbb{R}^n$

$$\mathbb{R}^n = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \ \underbrace{u_1, u_2, .., u_n}_{\text{real numbers}} \in \mathbb{R} \right\}$$

Zero vector, $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

Algebraic Properties of $\mathbb{R}^n$

For all **u**, **v**, **w** in $\mathbb{R}^n$ and all scalars $c$ and $d$:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$,
where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vii) $c(d\mathbf{u}) = (cd)(\mathbf{u})$

(viii) $1\mathbf{u} = \mathbf{u}$

For simplicity of notation, a vector such as $\mathbf{u} + (-1)\mathbf{v}$ is often written as $\mathbf{u} - \mathbf{v}$. Figure 7 shows $\mathbf{u} - \mathbf{v}$ as the sum of $\mathbf{u}$ and $-\mathbf{v}$.

# Linear Combinations

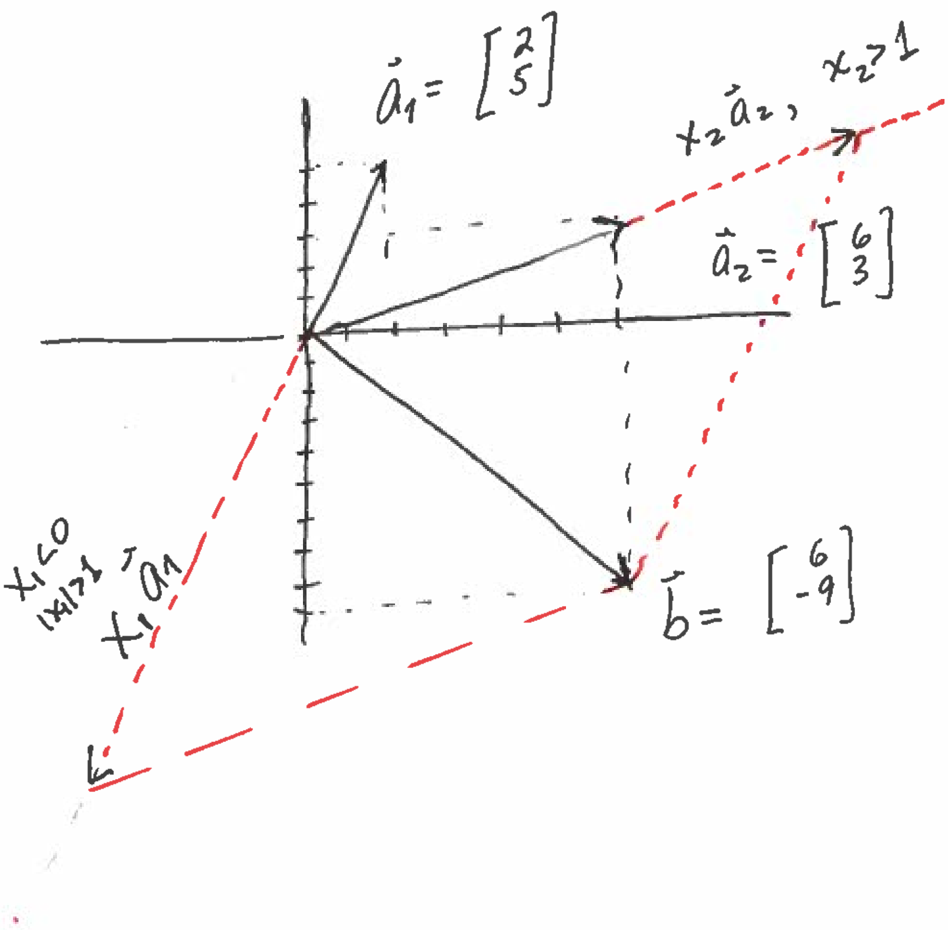
Consider $\vec{v}_1, .., \vec{v}_p$ in $\mathbb{R}^n$ and Scalars (real numbers) $c_1, c_2, .., c_p$, then

$$\vec{y} = c_1 \vec{v}_1 + \cdots + c_p \vec{v}_p$$

is Called a linear combination of $\vec{v}_1, .., \vec{v}_p$ with weights $c_1, c_2, .., c_p$.

__Exercise__ Determine if $\vec{b} = \begin{bmatrix} 6 \\ -9 \end{bmatrix}$ is a linear combination of $\vec{a}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\vec{a}_2 = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

__answ__:



We want to find weights $x_1$ and $x_2$ such that

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 = \vec{b} \quad \text{or}$$

$$x_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -9 \end{bmatrix}$$

This is equivalent to solve the linear system

$$\begin{cases} 2x_1 + 6x_2 = 6 \\ 5x_1 + 3x_2 = -9 \end{cases}$$

whose <u>augmented matrix</u> is given by

$$\begin{bmatrix} 2 & 6 & 6 \\ 5 & 3 & -9 \end{bmatrix} \overset{\text{Short notation}}{=\!=} \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{b} \end{bmatrix}$$

By row reducing it

$$\begin{bmatrix} 2 & 6 & 6 \\ 5 & 3 & -9 \end{bmatrix} \overset{\frac{\text{row1}}{2}}{\sim} \begin{bmatrix} 1 & 3 & 3 \\ 5 & 3 & -9 \end{bmatrix}$$

$$\overset{(-5)\text{row1} + \text{row2}}{\sim} \begin{bmatrix} 1 & 3 & 3 \\ 0 & -12 & -24 \end{bmatrix} \Rightarrow \begin{array}{l} -12x_2 = -24 \\ \Rightarrow \boxed{x_2 = 2} \end{array}$$

and

$$x_1 = 3 - 3x_2$$
$$= 3 - 6$$
$$\Rightarrow \boxed{x_1 = -3}$$

It means

1) $\begin{bmatrix} 6 \\ -9 \end{bmatrix} = -3 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 2 \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

$$\vec{b} = -3\,\vec{a}_1 + 2\,\vec{a}_2 .$$

$\vec{b} = \begin{bmatrix} 6 \\ -9 \end{bmatrix}$ is a linear combination of

$\vec{a}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\vec{a}_2 = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$.

2) The linear system

$$\begin{cases} 2x_1 + 6x_2 = 6 \\ 5x_1 + 3x_2 = -9 \end{cases}$$

has a solution $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

This solution is unique, why?

Question:

Can any $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$ be obtained as a linear combination of $\vec{a}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\vec{a}_2 = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$?

Let's see. We want to know if there are $x_1$ and $x_2$ such that

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 = \vec{b} \text{ or}$$

$$x_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Proceeding as in the previous problem: we consider the corresponding augmented matrix

$$\begin{bmatrix} 2 & 6 & b_1 \\ 5 & 3 & b_2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & b_1/2 \\ 5 & 3 & b_2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & b_1/2 \\ 0 & -12 & -\frac{5b_1}{2} + b_2 \end{bmatrix}$$

$$\Rightarrow \quad x_2 = \frac{5}{24} b_1 - \frac{b_2}{12} \Rightarrow \boxed{x_2 = \frac{1}{12}\left[\frac{5}{2} b_1 - b_2\right]}$$

$$x_1 = \frac{b_1}{2} - 3x_2 = \frac{b_1}{2} - 3\left(\frac{5}{24} b_1 - \frac{b_2}{12}\right)$$

or $\boxed{x_1 = \frac{-1}{8} b_1 + \frac{b_2}{4}}$

4.1

One of the key ideas in linear algebra is to study the set of all vectors that can be generated or written as a linear combination of a fixed set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of vectors.

DEFINITION

If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated) by** $\mathbf{v}_1, \dots, \mathbf{v}_p$. That is, Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with $c_1, \dots, c_p$ scalars.

Asking whether a vector $\mathbf{b}$ is in Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ amounts to asking whether the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$$

has a solution, or, equivalently, asking whether the linear system with augmented matrix $[\,\mathbf{v}_1 \;\; \cdots \;\; \mathbf{v}_p \;\; \mathbf{b}\,]$ has a solution.

Note that Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ contains every scalar multiple of $\mathbf{v}_1$ (for example), since $c\mathbf{v}_1 = c\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p$. In particular, the zero vector must be in Span $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

## A Geometric Description of Span $\{\mathbf{v}\}$ and Span $\{\mathbf{u}, \mathbf{v}\}$

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$. Then Span $\{\mathbf{v}\}$ is the set of all scalar multiples of $\mathbf{v}$, which is the set of points on the line in $\mathbb{R}^3$ through $\mathbf{v}$ and $\mathbf{0}$. See Fig. 10.

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^3$, with $\mathbf{v}$ not a multiple of $\mathbf{u}$, then Span $\{\mathbf{u}, \mathbf{v}\}$ is the plane in $\mathbb{R}^3$ that contains $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{0}$. In particular, Span $\{\mathbf{u}, \mathbf{v}\}$ contains the line in $\mathbb{R}^3$ through $\mathbf{u}$ and $\mathbf{0}$ and the line through $\mathbf{v}$ and $\mathbf{0}$. See Fig. 11.

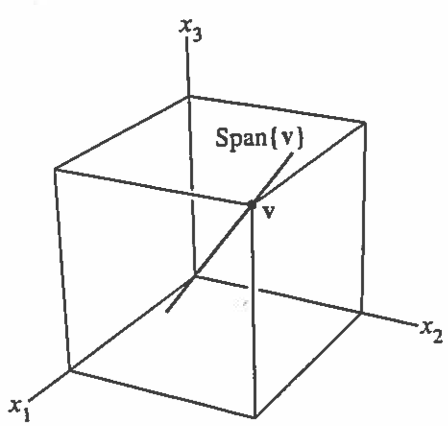



**FIGURE 10** Span $\{\mathbf{v}\}$ as a line through the origin.

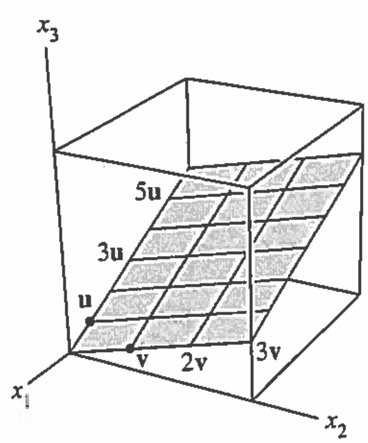
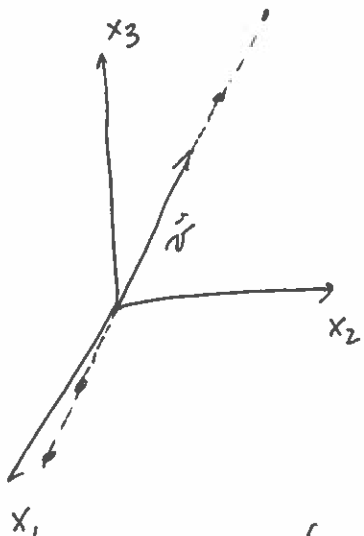



**FIGURE 11** Span $\{\mathbf{u}, \mathbf{v}\}$ as a plane through the origin.

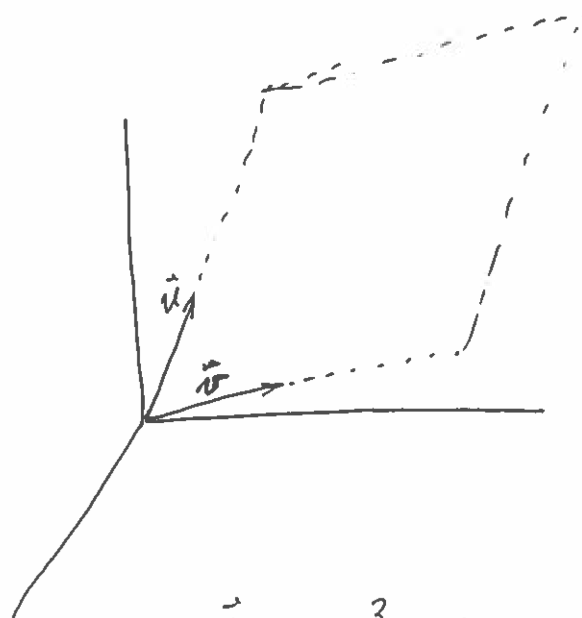
# Geometric Description of Span $\{\vec{v}\}$ and Span $\{\vec{u}, \vec{v}\}$ in $\mathbb{R}^3$




$$\text{Span}\{\vec{v}\} = \left\{ \vec{x} \text{ in } \mathbb{R}^3, \text{ such that } \vec{x} = c\vec{v} \quad c \text{ is real} \right\}$$

Line thru origin

$$\text{Span}\{\vec{u}, \vec{v}\} = \left\{ \vec{x} \text{ in } \mathbb{R}^3, \text{ such that } \vec{x} = c\vec{u} + d\vec{v}, \text{ where } c, d \text{ are real numbers} \right\}$$




Plane in $\mathbb{R}^3$ containing $\vec{u}$ and $\vec{v}$.
if $\vec{u} \neq \alpha \vec{v}$

__Question:__ Given a vector $\vec{b}$ in $\mathbb{R}^3$, How can we determine if $\vec{b}$ is in the plane generated by $\vec{u}$ and $\vec{v}$.

Questions:

1) What can be said about

$$\text{Span}\left\{ \underset{\vec{a}_1}{\begin{bmatrix} 2 \\ 5 \end{bmatrix}}, \underset{\vec{a}_2}{\begin{bmatrix} 6 \\ 3 \end{bmatrix}} \right\} \quad ?$$

2) What can be said about span $\{\vec{a}_1, \vec{a}_2\}$ for arbitrary vectors $\vec{a}_1$ and $\vec{a}_2 \in \mathbb{R}^2$?

3) What can be said about span $\{\vec{a}_1, \vec{a}_2, .., \vec{a}_p\}$ $p \geq 3$ in $\mathbb{R}^2$?

4) What is the minimum number of vectors in $\mathbb{R}^2$ needed to span $\mathbb{R}^2$?

5) How would you extend these observations to $\mathbb{R}^3$?

Consider now the linear system

$$\begin{cases} 2x_1 + 6x_2 = b_1 \\ 5x_1 + 3x_2 = b_2 \end{cases}$$

i) Does this linear system have a solution for any $b_1$ and $b_2$? Why?

ii) If the answer is yes, is this solution unique? Why?

iii) In the previos row-reduction process, What is the echelon form of the coefficient matrix

$$A = \begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}.?$$

Does it depend on $\vec{b}$?

## Conclusions:

i) A vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$$

has the same set of solutions as the linear system whose augmented matrix is

$$\left[\vec{a}_1 \;\; \vec{a}_2 \; \ldots \; \vec{a}_n \;\; \vec{b}\right] \qquad (4.1)$$

ii) $\vec{b}$ can be generated as a linear combination of $\vec{a}_1, \ldots, \vec{a}_n$ if and only if there exists a solution to the linear system corresponding to (4.1).

## Exercise 28

two types of coal A, B.

$$X_1 \begin{bmatrix} 27.6 \text{ h} \\ 3100 \text{ sd} \\ 250 \text{ p.m} \end{bmatrix} + X_2 \begin{bmatrix} 30.2 \text{ h} \\ 6400 \text{ sd} \\ 360 \text{ p.m} \end{bmatrix} = \begin{bmatrix} \text{Total} \\ \text{production} \\ \text{of} \\ \text{each} \end{bmatrix} \begin{matrix} \text{h} \\ \text{sd} \\ \text{p.m} \end{matrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \vec{C}$$

($X_1$ = tons A, $X_2$ = tons B)

a) Heat produced is $27.6X_1 + 30.2X_2$ million Btu

b)
$$\begin{cases} 27.6\, X_1 + 30.2\, X_2 = C_1 & \text{million Btu} \\ 3100\, X_1 + \quad X_2 = C_2 & \text{gr. s.d} \\ 250\, X_1 + 360\, X_3 = C_3 & \text{gr. p.m} \end{cases}$$

If we know the end result after burning, we may want to estimate
How much tons of A and B were used?

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Also, we may want to know: how much of A and B are necessary to produce certain given amounts of Btu heat
S.d.
and p.m.