7.1
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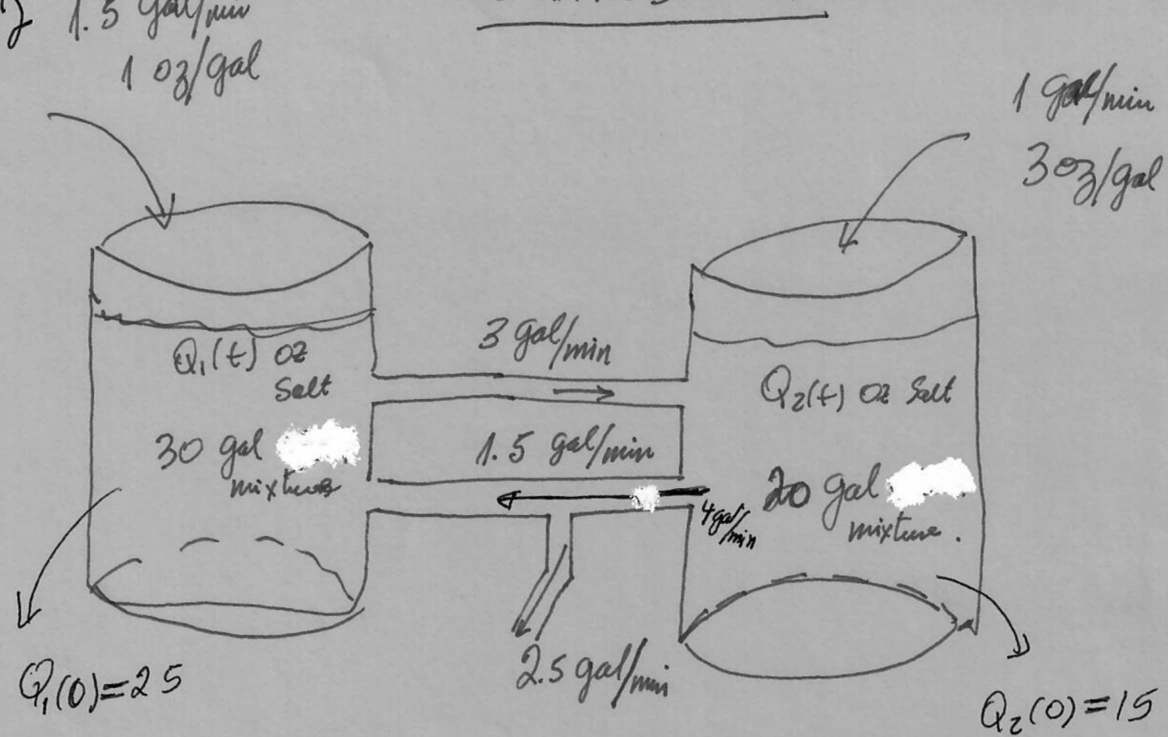
7.1 INTRODUCTION.



$Q_1(t)$: Amount of Salt at time t. (oz) in Tank 1.

$Q_2(t)$: " " " " " in Tank 2.

$$\begin{cases} \dfrac{dQ_1}{dt} = 1\,\dfrac{oz}{gal} * 1.5\,\dfrac{gal}{min} - \dfrac{Q_1(t)\,oz}{30\,gal} * 3\,\dfrac{gal}{min} + \dfrac{Q_2(t)\,oz}{20\,gal} * 1.5\,\dfrac{gal}{min}. \\[2ex] \dfrac{dQ_2}{dt} = 3\,\dfrac{oz}{gal} * 1\,\dfrac{gal}{min} - \dfrac{Q_2(t)\,oz}{20\,gal} * 4\,\dfrac{gal}{min} + \dfrac{Q_1(t)\,oz}{30\,gal} * 3\,\dfrac{gal}{min} \end{cases}$$

So

$$\begin{cases} \dfrac{dQ_1}{dt} = 1.5 - \dfrac{Q_1(t)}{10} + \dfrac{3}{40} Q_2(t) \\[2ex] \dfrac{dQ_2}{dt} = 3 - \dfrac{Q_2(t)}{5} + \dfrac{Q_1(t)}{10} \end{cases}$$

Linear system of first order ODEs
Non homogeneous.

$Q_1(0) = 25$, $Q_2(0) = 15$.

First order Systems of Linear differential Equations

Written as a matrix equation.

Tanks Problem:

$$
\begin{cases}
\dfrac{dQ_1}{dt} = -\dfrac{1}{10} Q_1 + \dfrac{3}{40} Q_2 + 1.5 & (1.1) \\[2ex]
\dfrac{dQ_2}{dt} = \dfrac{1}{10} Q_1 - \dfrac{1}{5} Q_2 + 3 & (1.2) \\[2ex]
Q_1(0) = 25, \qquad Q_2(0) = 15. & (1.3)
\end{cases}
$$

Define

$$\vec{Q}(t) = \begin{pmatrix} Q_1(t) \\ Q_2(t) \end{pmatrix}$$

then, (1.1)-(1.3) can be written as

$$\vec{Q}'(t) = \begin{pmatrix} Q_1'(t) \\ Q_2'(t) \end{pmatrix} = \overbrace{\begin{pmatrix} -\frac{1}{10} & \frac{3}{40} \\ \frac{1}{10} & -\frac{1}{5} \end{pmatrix}}^{=A} \overbrace{\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}}^{=\vec{Q}} + \overbrace{\begin{pmatrix} 1.5 \\ 0 \end{pmatrix}}^{=\vec{G}(t)}$$

or

$$\boxed{Q'(t) = A\,\vec{Q}(t) + \vec{G}(t)}$$

CHAPTER 7

Systems of First Order Linear Equations.

Exercise 2 (Sect 7.1).

Transform the 2nd order ODE:

$$u'' + 0.5u' + 2u = 3\sin t \qquad (1)$$

into a system of first order ODEs.

Define $x_1 = u$, $x_2 = u'$

then $x_1' = x_2$ and $x_2' = u''$

Subst. into (1)

$$x_2' + 0.5x_2 + 2x_1 = 3\sin t$$

or

$$\begin{cases} x_1' = x_2 \\ x_2' = -0.5x_2 - 2x_1 + 3\sin t \end{cases}$$

First order linear system of ODEs Non-homogeneous.

In General, an arbitrary nth order equ.

$$y^{(n)} = F(t, y, y', \ldots, y^{(n-1)})$$

Can be written as a System of "n" 1st order equs.

$$\left.\begin{aligned} x_1 &= y \\ x_2 &= y' \\ &\vdots \\ x_{n} &= y^{(n-1)} \end{aligned}\right\} \Rightarrow \begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= F(t, x_1, x_2, \ldots, x_n) \end{aligned}$$

$$\Rightarrow x_n' = y^{(n)} = F(t, \ldots, y^{(n-1)})$$

Also, the two masses Spring-mass system of example 2 Can be written as a 4 1st order system of equations

7.1 probl. 2

$$u'' + 0.5u' + 2u = 3\sin t$$

$$x_1 = u \qquad x_2 = u' \qquad \boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= -2x_1 - 0.5x_2 + 3\sin t \end{aligned}}$$

$$x_1' = u' = x_2$$

$$x_2' = u'' = -0.5u' - 2u + 3\sin t = -0.5x_2 - 2x_1 + 3\sin t$$

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -0.5 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 3\sin t \end{pmatrix}$$

In particular, all vibration problems

$$my'' + \gamma y' + ky = F(t)$$

$$y'' = -\frac{k}{m}y - \frac{\gamma}{m}y' + \frac{F(t)}{m}$$

then if $x_1 = y$, $x_2 = y'$

$$\Rightarrow x_1' = y' = x_2$$

$$x_2' = y'' = -\frac{k}{m}y - \frac{\gamma}{m}y' + \frac{F(t)}{m} = -\frac{k}{m}x_1 - \frac{\gamma}{m}x_2 + \frac{F(t)}{m}$$

$$\therefore \left.\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{k}{m}x_1 - \frac{\gamma}{m}x_2 + \frac{F(t)}{m} \end{aligned}\right\} \Leftrightarrow \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ F(t)/m \end{pmatrix}$$

General system of 1st order equations.

$$\text{IVP}\begin{cases} X_1'(t) = F_1(t, X_1, .., X_n) \\ X_2'(t) = F_2(t, X_1, \ldots, X_n) \\ \vdots \\ X_n'(t) = F_n(t, X_1, .., X_n) \\ \text{I.C's: } X_1(t_0) = X_{10}, \ldots, \quad X_n(t_0) = X_{n0}. \end{cases}$$

Extension 1st order ODE IVP

$$\text{IVP}\begin{cases} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

Exist. and uniqueness thm:

If i) $f(t,y)$ and $\frac{\partial f}{\partial y}(t,y)$ are continuous in rectangle R: $(\alpha, \beta) \times (\gamma, \delta)$ of the ty-plane.

ii) $(t_0, y_0) \in R$.

then There is a unique solution $y(t)$ to the IVP defined on the interval $(t_0 - h, t_0 + h) \subset (\gamma, \delta)$.

Extension to systems of ODE.

If i) $F_i(t, x_1, .., x_n)$ and $\dfrac{\partial F_i}{\partial x_j}$, $\begin{matrix} i=1,..,n \\ j=1,..,n \end{matrix}$

are continuous on the region R of $tx_1..x_n$-space defined by $\alpha < t < \beta$, $\alpha_1 < x_1 < \beta_1$, .., $\alpha_n < x_n < \beta_n$.

ii) $(t_0, x_{10}, x_{20}, .., x_{n0}) \in R$.

Then there is a unique solution

$x_1(t), .., x_n(t)$ for the IVP defined on

the interval: $(t_0 - h, t_0 + h) \subset (\alpha, \beta)$.

Linear Systems of ODE.

$$\text{IVP}\begin{cases} x_1'(t) = p_{11}(t)\,x_1(t) + \cdots + p_{1n}(t)\,x_n(t) + g_1(t) \\ x_2'(t) = p_{21}(t)\,x_1(t) + \cdots + p_{2n}(t)\,x_n(t) + g_2(t) \\ \vdots \\ x_n'(t) = p_{n1}(t)\,x_1(t) + \cdots + p_{nn}(t)\,x_n(t) + g_n(t). \\ \text{I.C's: } x_1(t_0) = x_{1_0}, \ldots, \quad x_n(t_0) = x_{n_0} \end{cases}$$

Discuss: Homogeneous, Non-homogeneous, Constant and variable coefficients.

Matrix Notation:

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} = \begin{pmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & & \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix} \qquad (*)$$

or

$$\text{IVP.}\begin{cases} \vec{x}\,'(t) = P(t)\,\vec{x}(t) + \vec{g}(t) & (5.1) \\ \vec{x}(t_0) = \vec{x}_0 = (x_{1_0}, \ldots, x_{n_0}). & (5.2) \end{cases}$$

(II) Thm. – (Existence and uniqueness)

If i) $p_{11}(t)$.., $p_{1n}(t)$.., $p_{n1}(t)$., $p_{nn}(t)$ are conts on (α,β) and $g_1(t)$, .., $g_n(t)$ are also conts on (α,β)

ii) $t_0 \in (\alpha,\beta)$

Then there is a unique solution for the IVP.(5) defined on (α,β).

(I) Def – A vector function $\vec{X}(t) = \vec{\phi}(t) = (\phi_1(t),.., \phi_n(t))$ is a solution of $(*)$ on (α,β) if its components satisfy the system of equations $(*)$ for all $t \in (\alpha,\beta)$.

7.4 Basic Theory of 1st order Linear Systems.

Thm 7.4.1 (Superposition pple)

If $\vec{X}^{(1)}(t)$ and $\vec{X}^{(2)}(t)$ are solns. of

$$\vec{X}'(t) = P(t)\vec{X}(t) \qquad (5.3)$$

where $P(t)$ has all entries conts. on (α,β)

then $\vec{X}^1(t) + \vec{X}^2(t)$ is also a soln. of (5.3)

Proof – based on linearity.

Linear independence.

The vector functions $\vec{X}^{(1)}(t), .., \vec{X}^{(n)}(t)$ are linearly independent on (α, β) if

$$C_1 \vec{X}^{(1)}(t) + C_2 \vec{X}^{(2)}(t) + \cdots + C_n \vec{X}^{(n)}(t) = 0(t), \quad t \in (\alpha, \beta)$$

$$\Rightarrow \quad C_1 = 0, \ldots, C_n = 0.$$

Extension of the definition of Wronskian for systems of ODE.

Ex. # 5 (7.5) Consider

$$\vec{x}_1'(t) = -2x_1(t) + x_2(t) \qquad (1.1)$$

$$\vec{x}_2'(t) = x_1(t) - 2x_2(t) \qquad (1.2)$$

or

$$\vec{x}'(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Two solutions are

$$\vec{x}^{(1)}(t) = \begin{pmatrix} e^{-3t} \\ -e^{-3t} \end{pmatrix}, \qquad \vec{x}^{(2)}(t) = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix} \qquad (1.3).$$

Consider the matrix $X(t)$ of the two vector solutions.

$$X(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix} = \begin{pmatrix} \vec{X}^{(1)}(t) & \vec{X}^{(2)}(t) \\ \downarrow & \downarrow \end{pmatrix}$$

From elem. linear algebra, the column vectors $\vec{X}^{(1)}(t_0)$ and $\vec{X}^{(2)}(t_0)$ are linearly indep. if and only if $\det[X(t_0)] \neq 0$. $\underline{t = t_0}$.

Def.:- Consider the matrix $X(t)$ whose column vectors are vector functions solutions or not of the homogeneous linear system (5.3)

$$X(t) = \begin{pmatrix} X_{11}(t) & \cdots & X_{1n}(t) \\ \vdots & & \vdots \\ X_{n1}(t) & & X_{nn}(t) \end{pmatrix} = \begin{pmatrix} \vec{X}^{(1)}(t) & \cdots & \vec{X}^{(n)}(t) \\ \downarrow & & \downarrow \end{pmatrix}$$

then, the determinant

$\underline{\det(X(t))}$ is called the Wronskian of the vector functions $\vec{X}^{(1)}(t), \ldots, \vec{X}^{(n)}(t)$ and is denoted by

$$\underline{W[\vec{X}^{(1)}, \ldots, \vec{X}^{(n)}](t)}.$$

<u>Thm.</u>- Consider a set of vector functions

$\{\vec{X}^{(1)}(t),.., \vec{X}^{(n)}(t)\}$ defined on (α,β)

If $W[\vec{x}^{(1)},.., \vec{x}^{(n)}](t_0) \neq 0$ for any $t_0 \in (\alpha,\beta)$

then $\{\vec{X}^{(1)}(t),.., \vec{X}^{(n)}(t)\}$ are linearly indep. on (α,β).

<u>Proof.</u>- From elem. linear algebra

If $W[\vec{x}^{(1)},.., \vec{x}^{(n)}](t_0) \neq 0 \Rightarrow$ The vectors $\vec{X}^{(1)}(t_0) .., \vec{X}^{(n)}(t_0)$ are linearly indep. $(*)$

Therefore, if

$$C_1 \vec{X}^{(1)}(t) + .. + C_n \vec{X}^{(n)}(t) = 0(t), \quad \text{for all } t \in (\alpha,\beta)$$

in particular for $t = t_0$

$$C_1 \vec{X}^{(1)}(t_0) + \cdots + C_n \vec{X}^{(n)}(t_0) = 0 \overset{(*)}{\Longrightarrow} C_1 = 0,.., C_n = 0$$

$\Rightarrow \vec{X}^{(1)}(t).. \vec{X}^{(n)}(t)$ are lin. indep on (α,β).

For $\vec{X}^{(1)}(t),.., \vec{X}^{(n)}(t)$ solutions of (5.3), we have a stronger result.

<u>Thm 7.4.3</u> If $\vec{X}^{(1)}(t),.., \vec{X}^{(n)}(t)$ are solns. of (5.3) on (α,β)

then $W[\vec{X}^{(1)},.., \vec{X}^{(n)}](t) = 0$, for all $t \in (\alpha,\beta)$

or $W[\vec{X}^{(1)},.., \vec{X}^{(n)}](t) \neq 0$, for all $t \in (\alpha,\beta)$.

turn

Theorem 7.4.2

If the vector functions $\vec{x}^{(1)}(t), .., \vec{x}^{(n)}(t)$ are linearly independent solutions of the system

$$\vec{x}'(t) = A(t)\,\vec{x}(t) \qquad (6.1)$$

for each point in the interval $\alpha < t < \beta$, then each solution $\vec{x} = \vec{\phi}(t)$ of (6.1) can be expressed as a linear combination

$$\vec{\phi}(t) = c_1 \vec{x}^{(1)}(t) + \cdots + c_n \vec{x}^{(n)}(t)$$

in exactly one way (unique representation).

The set of solutions $\{\vec{x}^{(1)}(t), .., \vec{x}^{(n)}(t)\}$ of (6.1) is called a Fundamental set of solutions. for that interval.

In our example, $\vec{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ (6.2)

A fundamental set is formed by

$$\vec{x}^{(1)}(t) = \begin{pmatrix} e^{-3t} \\ -e^{-3t} \end{pmatrix}, \quad \vec{x}^{(2)}(t) = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$$

The general solution of (6.2) is given by

$$\vec{X}(t) = C_1 \vec{X}^{(1)}(t) + C_2 \vec{X}^{(2)}(t).$$

$$\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = C_1 \begin{pmatrix} X_1^{(1)}(t) \\ X_2^{(1)}(t) \end{pmatrix} + C_2 \begin{pmatrix} \vec{X}_1^{(2)}(t) \\ \vec{X}_2^{(2)}(t) \end{pmatrix}$$

$$= C_1 \begin{pmatrix} e^{-3t} \\ -e^{-3t} \end{pmatrix} + C_2 \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}.$$