6.6 Convolution Integral

We are interested in a formula for

$$\mathcal{L}^{-1}\{F(s)\,G(s)\}$$

Our first reaction may be to claim that

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = f(t)\,g(t)$$

Where $f(t) = \mathcal{L}^{-1}\{F(s)\}$ and $g(t) = \mathcal{L}^{-1}\{G(s)\}$

For example, if

$$F(s) = G(s) = \frac{1}{s-a}$$

then, we may be tempted to say that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\cdot\frac{1}{s-a}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}$$

$$= e^{at}\cdot e^{at} = e^{2at}. \quad ?$$

But, we know from thm. 6.3.2

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c)\,, \quad s > a+c \quad (2.1)$$

Then,

$$\mathcal{L}^{-1}\{F(s-c)\} = e^{ct}f(t). \qquad (2.2)$$

we also know, $\mathcal{L}\{t\} = \frac{1}{s^2}$ Table 6.2.1 #3.

Therefore, applying (2.2)

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\} = e^{at}t.$$

Thm 6.6.1

1) $F(s) = \mathcal{L}\{f(t)\}$
$G(s) = \mathcal{L}\{g(t)\}$, exist for $s > a \geq 0$

Then,
$$H(s) = F(s)\,G(s) = \mathcal{L}\{h(t)\}, \quad s > a$$

where
$$h(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

h is known as the convolution of f and g.

the integrals are called convolution integrals.

Short notation:
$$h(t) = f(t) * g(t).$$

Using thm 6.6.1

Find $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\} = h(t)$

Notice that $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} = f(t)$

Then $h(t) = \int_0^t f(t-\tau)\, f(\tau)\, d\tau$

$= \int_0^t e^{a(t-\tau)} e^{a\tau}\, d\tau$

$= \int_0^t e^{at-a\tau} \cdot e^{a\tau}\, d\tau$

$= \int_0^t e^{at} e^{(a\tau - a\tau)}\, d\tau = \int_0^t e^{at}\, d\tau$

$= e^{at}\,\tau \Big|_0^t = t e^{at}.$

or $\boxed{\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\} = t e^{at}}$ ✓ Check with Table 6.2.1 # 11.

Application of Convolution theorem to IVP with constant coefficients.

Consider
$$\begin{cases} ay'' + by' + cy = g(t) & (1.1) \\ y(0) = y_0, \quad y'(0) = y_0' & (1.2) \end{cases}$$

This problem can be separated into two IVPs.

In fact,
$$y(t) = y_1(t) + y_2(t)$$

where y_1 satisfies
$$\begin{cases} a y_1'' + by_1' + cy_1 = 0 & (1.3) \\ y_1(0) = y_0, \quad y_1'(0) = y_0' & (1.4) \end{cases}$$

and y_2 satisfies
$$\begin{cases} a y_2'' + by_2' + cy_2 = g(t) & (1.5) \\ y(0) = 0, \quad y'(0) = 0 & (1.6) \end{cases}$$

The IVP (1.1) - (1.2) is referred as

<u>input - Output problem</u>

where

① <u>Coefficients</u> <u>a, b, c</u> properties of Physical Sistem.

② <u>$g(t)$ is the input</u> to the system

③ y_0, y_0' describe the initial state

④ the Soln. $y(t)$ is the output at time t.

If we apply L.T. to these two problems, then we obtain

$$a\left(s^2 Y_1(s) - s y_0 - y_0'\right) + b\left(s Y_1(s) - y_0\right) + c Y_1(s) = 0$$

or

$$(as^2 + bs + c) Y_1(s) = (as + b) y_0 + a y_0'$$

Solving for $Y_1(s)$

$$\boxed{Y_1(s) = \frac{(as+b) y_0 + a y_0'}{as^2 + bs + c}}$$

$$\mathcal{L}\{a y_2'' + b y_2' + c y_2\} = \mathcal{L}\{g(t)\}$$

Also, or

$$(as^2 + bs + c) Y_2(s) = G(s)$$

Then,

$$\boxed{Y_2(s) = \frac{G(s)}{as^2 + bs + c}}$$

On the other hand, if we apply L.T. to (1.1) and (1.2)

$\mathcal{L}\{(1.1), (1.2)\}$

$$a\left(s^2Y(s) - sy_0 - y_0'\right) + b\left(sY(s) - y_0\right) + cY(s) = G(s)$$

Then, $(as^2 + bs + c)Y(s) = (as+b)y_0 + ay_0' + G(s)$

and

$$\boxed{\begin{aligned} Y(s) &= \underbrace{\frac{(as+b)y_0 + ay_0'}{as^2+bs+c}}_{Y_1(s)} + \underbrace{\frac{G(s)}{as^2+bs+c}}_{Y_2(s)} \\ \text{or}\quad Y(s) &= Y_1(s) + Y_2(s) \end{aligned}}$$

Then, to obtain $y(t)$, we can obtain $y_1(t)$ taking

(I) $\boxed{\mathcal{L}^{-1}\{Y_1(s)\} = y_1(t)}$

To do this, we use P.F.D. and Table 6.2.1.

and

II $\mathcal{L}^{-1}\{Y_2(s)\} = \mathcal{L}^{-1}\{H(s)\,G(s)\}$

where $H(s) = \dfrac{1}{as^2+bs+c}$.

is called transfer function. (properties of system)

Also, $G(s)$ depends on the external force $g(t)$.

Using convolution theorem

$$y_2(t) = \mathcal{L}^{-1}\{Y_2(s)\} = \mathcal{L}^{-1}\{H(s)\,G(s)\}$$

$$= \int_0^t h(t-\tau)\,g(\tau)\,d\tau$$

A physical interpretation of $h(t)$ can be obtained by considering

$$\begin{cases} a\hat{y}'' + b\hat{y}' + c\hat{y} = \delta(t) & (5.1) \\ \hat{y}(0) = 0, \quad \hat{y}'(0) = 0 & (5.2) \end{cases}$$

Response of the system to a unit impulse applied at $t=0$.

$\hat{y}(t)$: Impulse response of the system.

Applying $\mathcal{L}\{(5.1)-(5.2)\}$

$$(as^2 + bs + c)\,Y(s) = \mathcal{L}\{\delta(t)\} = \int_0^\infty e^{-st}\,\delta(t)\,dt = 1$$

$$\Rightarrow \quad Y(s) = \frac{1}{as^2+bs+c} = H(s)$$

$$\Rightarrow \quad h(t) = \hat{y}(t) : \text{Impulse response}$$

Then,

$$y_2(t) = \int_0^t h(t-\tau)\, g(\tau)\, d\tau$$

is the convolution of the ① impulse response $h(t)$ and ② the forcing function.

6.6 #16) $\begin{cases} y'' + y' + \frac{5}{4} y = 1 - u_\pi(t) \\ y(0) = 1, \quad y'(0) = -1 \end{cases}$

L.T.

$$\left(s^2 + s + \frac{5}{4}\right) Y(s) - s + 1 = \frac{1}{s} - \frac{e^{-\pi s}}{s} = G(s)$$

$$Y(s) = \frac{s-1}{s^2+s+5/4} + \frac{G(s)}{s^2+s+5/4} = Y_1(s) + Y_2(s)$$

Now, $$Y_1(s) = \frac{s-1}{s^2+s+5/4} = \frac{s-1}{(s+1/2)^2 - \frac{1}{4} + \frac{5}{4}} = \frac{s-1}{(s+1/2)^2 +}$$

$$= \frac{s}{(s+1/2)^2+1} - \frac{1}{(s+1/2)^2+1}$$

$$\Rightarrow \boxed{y_1(t) = e^{-t/2}\cos(t) - e^{-t/2}\sin(t)}$$

$$Y_2(s) = G(s)H(s) = \left(\frac{1}{s} - \frac{e^{-\pi s}}{s}\right)\left(\frac{1}{s^2+s+5/4}\right)$$

$$g(t) = 1 - u_\pi(t)$$

$$\Rightarrow \boxed{y_2(t) = \int_0^t \left(1 - u_\pi(\tau)\right)\left(e^{(t-\tau)/2} \sin(t-\tau)\right) d\tau}$$

$$\boxed{y(t) = y_1(t) + y_2(t)}$$

III

input: $f(t)$ —— Impulse response: $\hat{y}(t)$ —— Output: $y(t)$

$$y(t) = \hat{y}(t) * f(t)$$

$$Y(s) = \mathcal{L}\{y(t)\} = \mathcal{L}\{\downarrow\} = \hat{Y}(s)F(s)$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{\hat{Y}(s)F(s)\} = \int_0^t \hat{y}(t-\tau) f(\tau)\, d\tau.$$

Convenience ~~from~~ when input is changed (external force) but rhs of equation remains.

input: $F(s)$ $\longrightarrow$ System function: $\hat{Y}(s)$ $\longrightarrow$ output: $Y(s) = F(s)\hat{Y}(s)$.

What is the System function?

Calculation?

$$\begin{cases} y'' + 4y = \delta(t) - e^{-t} \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$$

$$(s^2+4)Y(s) - s\overset{=1}{y(0)} - \overset{=0}{y'(0)} = 1 - \frac{1}{s+1} = \frac{s}{s+1}$$

$$Y(s) = \left(s + \frac{s}{s+1}\right)\frac{1}{s^2+4}$$

$$Y(s) = \frac{s}{s^2+4} + \frac{s}{(s+1)(s^2+4)} = Y_1(s) + Y_2(s)$$

Now,

$$Y_2(s) = \frac{s}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Cs+D}{s^2+4}$$

$$= \frac{A(s^2+4) + Cs(s+1) + D(s+1)}{(s+1)(s^2+4)}$$

$$= \frac{(A+C)s^2 + (C+D)s + 4A + D}{(s+1)(s^2+4)}$$

Then,

s^2: $A + C = 0$

s: $C + D = 1$

1: $4A + D = 0$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 4 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -4 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 5 & 4 \end{pmatrix}$$

$\Rightarrow 5D = 4 \Rightarrow D = 4/5$

$C + D = 1 \Rightarrow C = -D + 1 = -4/5 + 1 = 1/5$

$A + C = 0 \Rightarrow A = -C = -1/5$

Then,

$$\frac{s}{(s+1)(s^2+4)} = \frac{-1/5}{s+1} + \frac{1/5\, s + 4/5}{s^2+4}$$

$$\frac{4}{5}\frac{1}{s^2+4} = \frac{4}{5}\frac{1}{2}\frac{2}{s^2+4}$$

$\mathcal{L}^{-1}$

$$\boxed{y_2(t) = -\frac{1}{5}e^{-t} + \frac{1}{5}\cos(2t) + \frac{\cancel{4}^{\,2}}{5}\frac{1}{\cancel{2}}\sin(2t)}$$

Also, $Y_1(s) = \dfrac{s}{s^2+4}$

$\mathcal{L}^{-1}$

$y_1(t) = \cos(2t)$

therefore, the solution of our IVP:

$$\boxed{y(t) = y_1(t) + y_2(t) = -\frac{1}{5}e^{-t} + \frac{2}{5}\sin(2t) + \frac{6}{5}\cos(2t)}$$

Let's call $g(t) = \delta(t) - e^{-t}$ (Input).

then, $\mathcal{L}\{g(t)\} = G(s) = 1 - \frac{1}{s+1} = \frac{s}{s+1}$

Also, Transfer function: $H(s) = \frac{1}{s^2+4} \xrightarrow{\mathcal{L}^{-1}} h(t) = \frac{1}{2}\sin(2t)$

Then,
$$Y_2(s) = H(s)\,G(s) = \frac{s}{(s^2+4)(s+1)}$$

Using Convolution theorem:

$$Y_2(s) \xrightarrow{\mathcal{L}^{-1}} y_2(t) = \int_0^t h(t-\tau)\,g(\tau)\,d\tau$$

$$= \frac{1}{2}\int_0^t \sin(2(t-\tau))\left(\delta(\tau) - e^{-\tau}\right)d\tau$$

$$= \frac{1}{2}\int_0^t \sin(2(t-\tau))\,\delta(\tau)\,d\tau - \frac{1}{2}\int_0^t e^{-\tau}\sin(2(t-\tau))\,d\tau$$

$$= \frac{1}{2}\int_0^t \sin(2\tau)\,\delta(t-\tau)\,d\tau - \frac{1}{2}\int_0^t e^{-(t-\tau)}\sin(2\tau)\,d\tau$$

properties of δ function

$$\frac{\sin(2t)}{2} - \frac{1}{2} e^{-t} \int_0^t e^{\tau} \sin(2\tau)\, d\tau$$

Int. by parts

$$\frac{\sin(2t)}{2} - \frac{e^{-t}}{2} \left[-\frac{2}{5} e^{t} \cos(2t) + \frac{1}{5} e^{t} \sin(2t) + \frac{2}{5} \right]$$

$$= \frac{\sin(2t)}{2} + \frac{1}{5} \cos(2t) - \frac{1}{10} \sin(2t) - \frac{1}{5} e^{-t}$$

or

$$\boxed{y_2(t) = \frac{2}{5} \sin(2t) + \frac{1}{5} \cos(2t) - \frac{1}{5} e^{-t}}$$