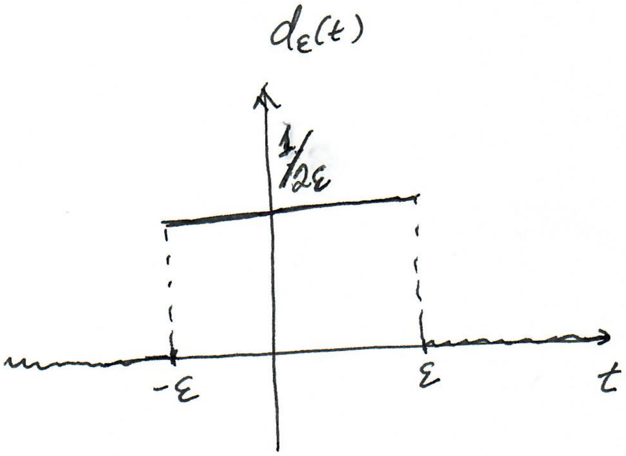
6.5 Impulse Function.

Forces of large magnitude that act over very short time intervals.(*)

Example:

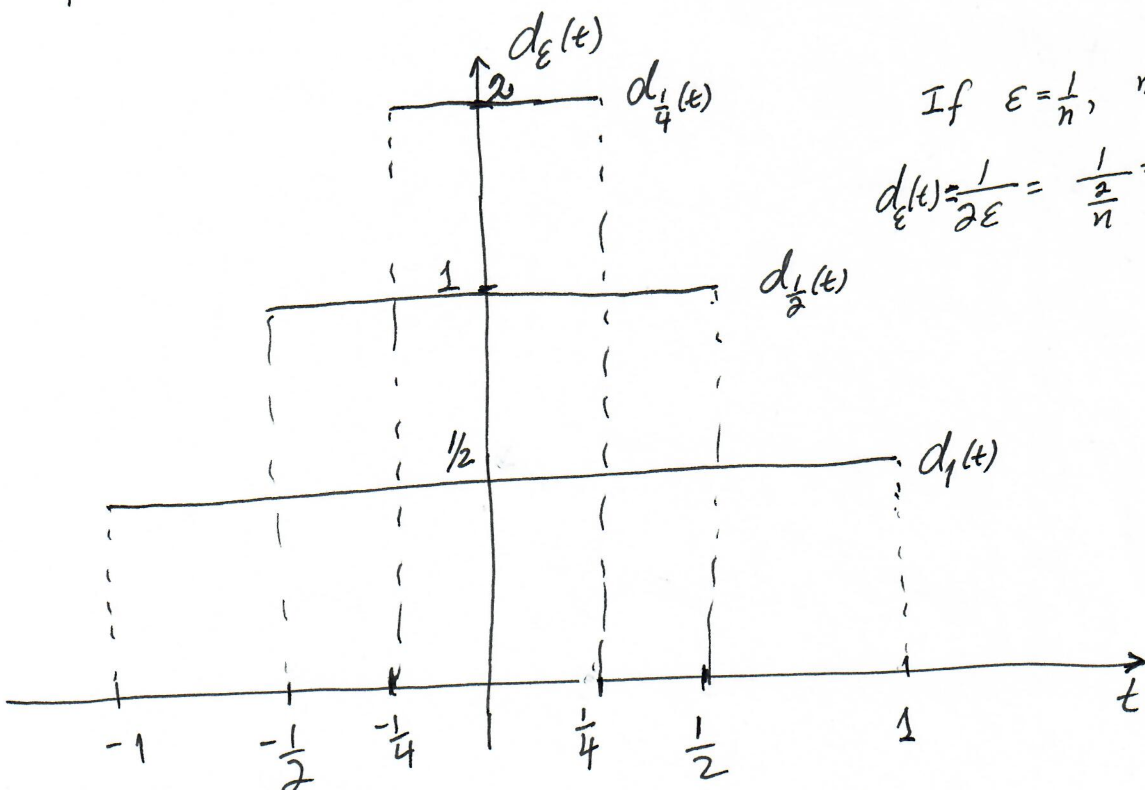
$$d_\varepsilon(t) = \begin{cases} 1/2\varepsilon, & -\varepsilon < t < \varepsilon \\ 0, & \text{otherwise} \end{cases}, \quad \varepsilon > 0.$$



$$I(\varepsilon) = \int_{-\infty}^{\infty} d_\varepsilon(t)\,dt =$$

$$= \int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon}\,dt = \frac{1}{2\varepsilon}\,t\Big|_{-\varepsilon}^{\varepsilon} = \frac{1}{2\varepsilon}(\varepsilon-(-\varepsilon)) = 1$$

for any $\varepsilon > 0$.



If $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$

$$d_\varepsilon(t) = \frac{1}{2\varepsilon} = \frac{1}{\frac{2}{n}} = \frac{n}{2}$$

(*) If the indep. variable were x (space) instead of time, then we talk about forces of large magnitude acting on very short space intervals

If $d_\varepsilon(t)$ represents a force, then

$$I(\varepsilon) = \int_{-\infty}^{\infty} d_\varepsilon(t)\,dt = \int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon}\,dt = 1, \quad \text{for any } \varepsilon > 0$$

represents the "total impulse" over the interval $(t_0-\varepsilon, t_0+\varepsilon)$.

Then, the functions $d_\varepsilon(t)$ have the following properties:

a) $\lim\limits_{\varepsilon \to 0} d_\varepsilon(t) = 0, \quad t \neq 0$

b) $\lim\limits_{\varepsilon \to 0} d_\varepsilon(0) = +\infty$

c) $\lim\limits_{\varepsilon \to 0} I(\varepsilon) = \lim\limits_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} d_\varepsilon(t)\,dt = \lim\limits_{\varepsilon \to 0} 1 = 1.$

Def. – An idealized unit impulse "function" $\delta(t)$, which imparts an impulse of magnitude one at $t=0$ and

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ +\infty, & t = 0 \end{cases}$$

and also satisfies the property:

If $f(t)$ is a continuous function on (α,β) and $0 \in (\alpha,\beta)$ then

$$\int_{-\infty}^{\infty} \delta(t) f(t)\,dt = f(0).$$

is called the Dirac delta function.

In particular, if $f(t) \equiv 1$, then

$$\int_{-\infty}^{\infty} \delta(t).1\,dt = \int_{-\infty}^{\infty} \delta(t)\,dt = 1 = \int_{-L}^{L} \delta(t)\,dt, \quad L>0.$$

So we have defined a "function" which is zero everywhere except at $t=0$ and whose integral over any interval containing zero is 1.

What is wrong with this function? SHOW WEBPAGE.

We can also have a unit impulse at any other point $t=t_0$. It means

$$\delta(t-t_0) = \begin{cases} 0, & t \neq t_0 \\ +\infty, & t = t_0 \end{cases} \qquad (3.1)$$

and

$$\boxed{\int_{-\infty}^{\infty} \delta(t-t_0)\, f(t)\,dt = f(t_0),} \qquad (3.2)$$

for any conts funct. $f(t)$ on (α, β) containing $t=t_0$.

In fact, $\delta(t-t_0)$ is normally defined by (3.2) functional operator.

Derivative of Heaviside fn. $\quad U_{t_0}(t) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$

$$\delta(t-t_0) = \frac{d}{dt} U_{t_0}(t) \quad \text{and} \quad U_{t_0}(t) = \int_{-\infty}^{t} \delta(\tilde{t}-t_0)\,d\tilde{t}$$

$\delta(t-t_0)$ is not p.w.c. and is not of exponential order. However, we can formally define its Laplace transform as usual for $t_0 > 0$

$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty e^{-st}\delta(t-t_0)\,dt = e^{-st_0}$$

In particular,

$$\mathcal{L}\{\delta(t)\} = \int_0^\infty e^{-st}\delta(t)\,dt = e^{-s.0} = 1.$$

Remark: $\delta(t-t_0)$ is not an ordinary function. However, working with impulsive problem we can formally operate on it as if it were an ordinary function. Justification rests on mathematical theory called "Theory of Distribution." Where functions are considered functional operators.

Exercise 6 (Sect. 6.5)

$$\begin{cases} y'' + 4y = \delta(t-4\pi) \\ y(0) = 1/2, \quad y'(0) = 0 \end{cases}$$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-4\pi)\}$$

$$s^2 Y(s) - s\underbrace{y(0)}_{=1/2} - \underbrace{y'(0)}_{=0} + 4Y(s) = e^{-s4\pi}$$

$$\Rightarrow \quad (s^2+4)Y(s) = e^{-4\pi s} + \frac{s}{2}$$

$$\Rightarrow \quad Y(s) = \frac{e^{-4\pi s}}{s^2+4} + \frac{s}{2(s^2+4)}$$

or $$Y(s) = \frac{e^{-4\pi s}}{2} \frac{2}{s^2+4} + \frac{1}{2}\frac{s}{s^2+4}$$

$$\downarrow \mathcal{L}^{-1}$$

Notice that $\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \sin(2t)$

and $\mathcal{L}^{-1}\left\{e^{-4\pi s} F(s)\right\} = u_{4\pi}(t) f(t-4\pi)$

Then,

$$\boxed{y(t) = \frac{1}{2} u_{4\pi} \sin(2(t-\pi/4)) + \frac{1}{2}\cos(2t)}$$

Math 334: Ordinary Differential Equations.
Section 6.5

Problem #10 Consider the initial value problem:

$$y'' + \gamma y' + y = \delta(t-1), \qquad y(0) = 0, \qquad y'(0) = 0. \tag{1}$$

If $\gamma = 1/2$, find the solution

Answer:

$$\mathcal{L}\{y'' + 1/2y' + y\} = \mathcal{L}\{\delta(t-1)\} = e^{-s}$$
$$s^2Y(s) - sy(0) - y'(0) + 1/2(sY(s) - y(0)) + Y(s) = e^{-s}$$

using ICs:

$$(s^2 + 1/2s + 1)Y(s) = e^{-s}, \quad \text{then} \quad Y(s) = \frac{e^{-s}}{s^2 + 1/2s + 1}$$

There is no need to use partial fraction decomposition. The next step is to complete the square in the denominator

$$Y(s) = \frac{e^{-s}}{(s+1/4)^2 + \frac{15}{16})} = \frac{e^{-s}}{(s+1/4)^2 + \frac{\sqrt{15}}{4})} = \frac{4}{\sqrt{15}}e^{-s}\frac{\frac{\sqrt{15}}{4}}{(s+1/4)^2 + \frac{\sqrt{15}}{4})}$$

or in more compact form

$$Y(s) = \frac{4}{\sqrt{15}}e^{-s}F(s), \quad \text{where} \quad F(s) = \frac{\frac{\sqrt{15}}{4}}{(s+1/4)^2 + \frac{\sqrt{15}}{4})}$$

Now, using the inverse Laplace transform

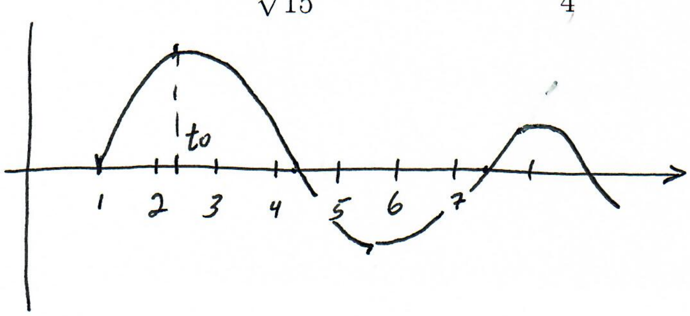
$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{4}{\sqrt{15}}\mathcal{L}^{-1}\{e^{-s}F(s)\} = \frac{4}{\sqrt{15}}u_1(t)f(t-1), \tag{2}$$

where

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{15}}{4}}{(s+1/4)^2 + \frac{\sqrt{15}}{4})}\right\} = e^{-\frac{1}{4}t}\sin\frac{\sqrt{15}}{4}t \tag{3}$$

Now, combining (2)-(3), we arrive to the solution (1) of:

$$y(t) = \frac{4}{\sqrt{15}}u_1(t)e^{-\frac{1}{4}(t-1)}\sin\frac{\sqrt{15}}{4}(t-1)$$



Max when

$y'(t_0) = 0$

For the provious inverse LT, we need

1) $\mathcal{L}^{-1}\left\{ \frac{b}{s^2+b^2} \right\} = \sin bt$

2) $\mathcal{L}^{-1}\left\{ \frac{b}{(s-a)^2+b^2} \right\} = e^{at} \sin bt.$

3) $\mathcal{L}^{-1}\left\{ e^{-cs} \frac{b}{(s-a)^2+b^2} \right\} = e^{a(t-c)} \sin b(t-c)\, u_c(t)$

Exercise 7 (6.5)

$$\begin{cases} y'' + y = \delta(t-2\pi)\cos t \\ y(0) = 0, \quad y'(0) = 1 \end{cases}$$

Notice that $\mathcal{L}\{\delta(t-t_0) f(t)\}$, $t_0 > 0$

$$= \int_0^\infty e^{-st} \delta(t-t_0) f(t)\, dt$$

$$= e^{-st_0} f(t_0)$$

Therefore, $\mathcal{L}\{\delta(t-2\pi)\cos t\} = e^{-2\pi s} \overset{=1}{\cos(2\pi)} = e^{-2\pi s}$

Finish this problem.

$$\mathcal{L}\{y''+y\} = e^{-2\pi s} \Leftrightarrow (s^2+1)Y(s) - s\overset{=0}{y(0)} - \overset{=1}{y'(0)} = e^{-2\pi s}$$

$$\Leftrightarrow \quad Y(s) = e^{-2\pi s}\frac{1}{s^2+1} + \frac{1}{s^2+1}$$

$$\Rightarrow y(t) = u_{2\pi}(t)\sin(t-2\pi) + \sin t$$