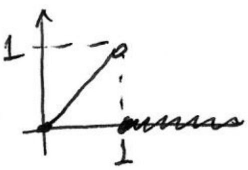
6.4 ODE's with discontinuous forcing functions.

Back to problem 25 (6.2)



$$y'' + y = \begin{cases} t, & 0 \le t \le 1 \\ 0, & 1 \le t < \infty \end{cases} \equiv f(t), \qquad \begin{array}{l} y(0)=0 \\ y'(0)=0 \end{array}$$

$\mathcal{L}\{f(t)\}$ was obtained in section 6.2 using the definition directly.

Now, we can represent $f(t)$ using step functions and then use thm. of 6.3. to obtain $\mathcal{L}\{f(t)\}$.

In fact,

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t - t u_1(t)\} = \mathcal{L}\{t\} - \mathcal{L}\{u_1(t) t\} =$$

$$= \mathcal{L}\{t\} - \mathcal{L}\{u_1(t)[(t-1)+1]\} =$$

$$= \mathcal{L}\{t\} - \mathcal{L}\{u_1(t)(t-1)\} - \mathcal{L}\{u_1(t)\} =$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} = \frac{1 - e^{-s} - s e^{-s}}{s^2} =$$

$$= \frac{1}{s^2} - \frac{e^{-s}(s+1)}{s^2}$$

On the other hand,

$$\mathcal{L}\{y''+y\} = s^2 Y(s) - s\overset{=0}{y(0)} - \overset{=0}{y'(0)} + Y(s)$$

$$= (s^2+1)\,Y(s)$$

$$\therefore \quad \boxed{Y(s) = \frac{1}{s^2(s^2+1)} - \frac{e^{-s}(s+1)}{s^2(s^2+1)}}$$

Partial fraction decomposition:

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1} = \frac{A(s^2+1)s + B(s^2+1) + s^2(Cs+D)}{s^2(s^2+1)}$$

$$= \frac{As^3 + As + Bs^2 + B + Cs^3 + Ds^2}{s^2(s^2+1)}$$

$$\begin{array}{llr}
s^3: & A + C = 0 & \boxed{C=0} \\
 & & \Uparrow \\
s^2: & B + D = 0 & \boxed{D=-1} \\
 & & \Uparrow \\
s: & A = 0 & \boxed{A=0} \\
s^0: & B = 1 \Rightarrow & \boxed{B=1}
\end{array}$$

$$\therefore \quad \frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

Similarly,

$$\frac{s+1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

s^3: $A + C = 0 \Rightarrow \boxed{C = -1}$

s^2: $B + D = 0 \Rightarrow \boxed{D = -1}$

s: $\boxed{A = 1}$

s^0: $\boxed{B = 1}$

$$\therefore \quad \frac{s+1}{s^2(s^2+1)} = \frac{1}{s} + \frac{1}{s^2} - \frac{s+1}{s^2+1}$$

$$\therefore \quad Y(s) = \frac{1}{s^2(s^2+1)} - \frac{e^{-s}(s+1)}{s^2(s^2+1)}$$

or $$Y(s) = \frac{1}{s^2} - \frac{1}{s^2+1} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \underbrace{\frac{e^{-s}(s+1)}{s^2+1}}_{e^{-s}\frac{s}{s^2+1} + e^{-s}\frac{1}{s^2+1}}$$

$\mathcal{L}^{-1}$

$$y(t) = t - \sin t - u_1(\cancel{t}) - u_1(t)(\cancel{t-1}) + u_1(t)\cos(t-1) + u_1(t)\sin(t-1)$$

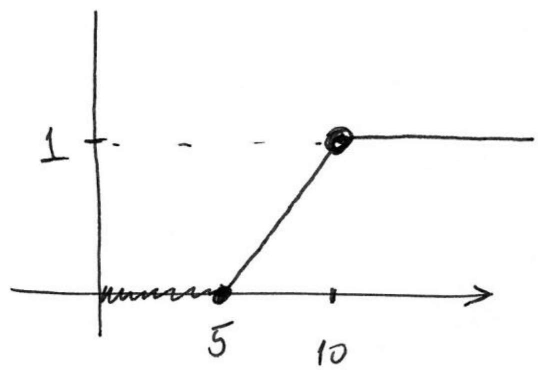
or

$$\boxed{y(t) = t - \sin t - t\,u_1(t) + u_1(t)\left[\cos(t-1) + \sin(t-1)\right].}$$

Example 2.- (book).

$$y'' + 4y = g(t) = \begin{cases} 0, & 0 \le t < 5 \\ \frac{t-5}{5}, & 5 \le t < 10 \\ 1, & t \ge 10 \end{cases} \qquad \begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$$

First, represent $g(t)$ using step. functions:



$$g(t) = U_5(t)\,\frac{t-5}{5} - U_{10}(t)\,\frac{t-5}{5} + U_{10}(t)$$

$$= U_5(t)\,\frac{t-5}{5} - U_{10}(t)\left[\frac{t-5}{5} - 1\right] =$$

$$= U_5(t)\,\frac{t-5}{5} - U_{10}(t)\left(\frac{t-10}{5}\right)$$

or

$$g(t) = \frac{1}{5}\left[U_5(t)(t-5) - U_{10}(t)(t-10)\right].$$

$$\Rightarrow \mathcal{L}\{g(t)\} = \frac{1}{5}\mathcal{L}\{U_5(t)(t-5)\} - \frac{1}{5}\mathcal{L}\{U_{10}(t)(t-10)\}$$

$$\Rightarrow G(s) = \frac{1}{5}\,e^{-5s}\frac{1}{s^2} - \frac{1}{5}\,e^{-10s}\frac{1}{s^2}$$

Applying Lapl. Transf. to the IVP:

$$\mathcal{L}\{y''+4y\} = \mathcal{L}\{g(t)\}$$

or $$s^2 Y(s) - s\overset{=0}{y(0)} - \overset{=0}{y'(0)} + 4Y(s) = G(s)$$

$$\Rightarrow \quad Y(s) = \frac{1}{5}\left\{\frac{e^{-5s}}{s^2(s^2+4)} - \frac{e^{-10s}}{s^2(s^2+4)}\right\}$$

If $$H(s) \equiv \frac{1}{s^2(s^2+4)}$$

then $$Y(s) = \frac{1}{5}\left\{e^{-5s}H(s) - e^{-10s}H(s)\right\}$$

$$\downarrow \mathcal{L}^{-1}$$

$$\Rightarrow \quad \boxed{y(t) = \frac{1}{5}\left[u_5(t)\,h(t-5) - u_{10}(t)\,h(t-10)\right]} \quad (*)$$

where $h(t) = \mathcal{L}^{-1}\{H(s)\}$.

Let's find $h(t)$

Solving linear syst. $A=0,\ B=1/4,\ C=0,\ D=-1/4$.

$$H(s) = \frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4} = \frac{1/4}{s^2} - \frac{1/4}{s^2+4}$$

$$\frac{1/4}{s^2+4} \rightarrow \frac{1}{8}\,\frac{2}{s^2+4}$$

$$\downarrow \mathcal{L}^{-1}$$

$$\Rightarrow \quad h(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

$$\Rightarrow h(t-5) = \frac{1}{4}(t-5) - \frac{1}{8}\sin(2(t-5)).$$

$$h(t-10) = 1/4\,(t-10) - \frac{1}{8}\sin(2(t-10))$$

Subst. into (*)

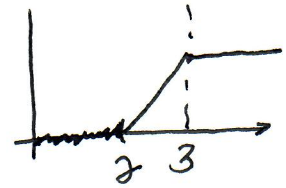
$$\boxed{\begin{aligned} y(t) = \frac{1}{20}&\left\{ u_5(t)\,(t-5) - u_{10}(t)\,(t-10) \right\} \\ &- \frac{1}{40}\left\{ u_5(t)\sin(2(t-5)) - u_{10}(t)\sin(2(t-10)) \right\} \end{aligned}}$$

$$\begin{cases} y''+2y'+10y = u_2(t)(t-2) - \overbrace{\underbrace{u_3(t)(t-2)+u_3(t)}_{-u_3(t)(t-3)}}^{-\left(u_3(t)(t-2)-u_3(t)\right)} \\ y(0)=0, \quad y'(0)=0 \end{cases}$$

$\mathcal{L}$:

$$s^2Y(s) - s\overset{=0}{y(0)} - \overset{=0}{y'(0)} + 2\left(sY(s)-y(0)\right) + 10Y(s)$$

$$= e^{-2s}\frac{1}{s^2} - \frac{e^{-3s}}{s^2}$$



or

$$\left(s^2+2s+10\right)Y(s) = \frac{e^{-2s}}{s^2} - \frac{e^{-3s}}{s^2}$$

Soln:

$$Y(s) = \frac{e^{-2s}}{s^2(s^2+2s+10)} - \frac{e^{-3s}}{s^2(s^2+2s+10)}$$

or

$$Y(s) = e^{-2s}F(s) - e^{-3s}F(s)$$

where $F(s) = \dfrac{1}{s^2(s^2+2s+10)} \xrightarrow{\mathcal{L}^{-1}} f(t) = ?$

$\mathcal{L}^{-1}$

$$\boxed{y(t) = u_2(t)\,f(t-2) - u_3(t)\,f(t-3)}$$

The problem reduces to find

$$f(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s^2(s^2+2s+10)} \right\}$$

Procedure:

1. Partial fraction decomposition
2. Modify fraction (using algebra) to transform them in expressions that are the Laplace transform of known functions.
3. Apply the inverse Laplace transform to obtain the solution.

In fact,

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$$F(s) = \frac{1}{s^2(s^2+2s+10)} = \underbrace{\frac{A}{s} + \frac{B}{s^2}} + \frac{Cs+D}{s^2+2s+10}$$

$$= \frac{As+B}{s^2} + \frac{Cs+D}{s^2+2s+10} = \frac{(As+B)(s^2+2s+10)+(Cs+D)s^2}{s^2(s^2+2s+10)}$$

$$= \frac{(A+C)s^3+(2A+B+D)s^2+(10A+2B)s^1+10B}{s^2(s^2+2s+10)}$$

$$\begin{cases} s^3: & A + C = 0 & \Rightarrow C = -A = 1/50 \\ s^2: & 2A + B + D = 0 & \Rightarrow D = -2A - B = +\frac{2}{50} - \frac{1}{10} = \frac{-3}{50} \\ s: & 10A + 2B = 0 & \Rightarrow A = \frac{-2B}{10} = -\frac{\cancel{2}^1}{\cancel{10}_5}\left(\frac{1}{10}\right) = -\frac{1}{50} \\ 1: & 10B = 1 & \Rightarrow B = 1/10 \end{cases}$$

$$\boxed{A = -1/50,\ B = 1/10,\ C = 1/50,\ D = \frac{-3}{50}}$$

Then,

$$F(s) = \frac{1}{s^2(s^2+2s+10)} = -\frac{1}{50}\frac{1}{s} + \frac{1}{10}\frac{1}{s^2} + \frac{1}{50}\left(\frac{s-3}{s^2+2s+10}\right)$$

Special work with

$$\frac{s-3}{s^2+2s+10} = \frac{s-3}{(s+1)^2+9} = \frac{s+1}{(s+1)^2+9} - \frac{4}{(s+1)^2+9}$$

$$= \frac{s+1}{(s+1)^2+9} - \frac{4}{3}\frac{3}{(s+1)^2+9}$$

then,

$$F(s) = \frac{1}{s^2(s^2+2s+10)} = -\frac{1}{50}\frac{1}{s} + \frac{1}{10}\frac{1}{s^2} + \frac{1}{50}\frac{s+1}{(s+1)^2+9} \overset{-2/75}{\overbrace{- \frac{1}{50}\frac{4}{3}}} \frac{3}{(s+1)^2+9}$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$\boxed{f(t) = -\frac{1}{50} + \frac{1}{10}t + \frac{1}{50}e^{-t}\cos 3t - \frac{2}{75}e^{-t}\sin 3t.}$$

and

$$\boxed{y(t) = u_2(t)\, f(t-2) - u_3(t)\, f(t-3)}$$