6.2 Solution of Initial Value Problems Using Laplace Transforms.

Our motivation to study Laplace transform in Sect 6.1 was to define a new technique to solve IVP for linear higher order nonhomogeneous ODEs with constant coefficients. In particular, when the forcing function (right hand side of the equation) is piecewise continuous or is an impulse function.

We began solving using Laplace transform the IVP:

$$\begin{cases} y'' - y' - 6y = 0 & (1.1) \\ y(0) = 1, \quad y'(0) = -1 & (1.2) \end{cases}$$

whose soln is given by

$$\boxed{y(t) = \frac{1}{5} e^{3t} + \frac{4}{5} e^{-2t},}$$

which is obtained using the characteristic equ.

$$r^2 - r - 6 = 0 \qquad \left(\text{which results assuming } y(t) = e^{rt}\right)$$

To apply L.T. (Laplace Transform) to (1.1) and (1.2), we need to know

$$\mathcal{L}\{y''-y'-6y\} = \mathcal{L}\{y''\}-\mathcal{L}\{y'\}-6\mathcal{L}\{y\} = ?$$

Theorem 6.2.1

Hipothesis: ① f is Continous on $0 \le t \le A$, for any $A > 0$.

② f' is piecewise continuous on $0 \le t \le A$ for any $A > 0$.

③ f is ~~of~~ exponential order

$|f(t)| \le K e^{at}$, $t \geq M$, K, M positive

then,

① $\mathcal{L}\{f(t)\}$ exists for $s > a$.

② $$\boxed{\begin{aligned}\mathcal{L}\{f'(t)\} &= s\,\mathcal{L}\{f(t)\} - f(0) \\ &= sF(s) - f(0).\end{aligned}}$$

Corollary:- ① f, f' Cont. on $0 \le t \le A$, for any $A > 0$.

② f, f' of exponential order for the same a, K, M.

③ f'' piecewise conts. on $0 \le t \le A$, for any $A > 0$.

Then,

$$\mathcal{L}\{f''(t)\} = s^2 \mathcal{L}\{f(t)\} - s f(0) - f'(0).$$

Proof.-

$$\mathcal{L}\{f''(t)\} = \mathcal{L}\{(f'(t))'\} =$$

$$\underset{\text{for } f''(t)}{\overset{\text{thm } 6.2.1}{=\!=\!=}} s\,\mathcal{L}\{f'(t)\} - f'(0) \underset{\text{for } f'(t)}{\overset{\text{thm } 6.2.1}{=\!=\!=}} s\left[s\,\mathcal{L}\{f(t)\} - f(0)\right] - f'(0)$$

or

$$\boxed{\begin{aligned}\mathcal{L}\{f''(t)\} &= s^2 \mathcal{L}\{f(t)\} - s f(0) - f'(0) \\ &= s^2 F(s) - s f(0) - f'(0)\end{aligned}}$$

Corollary 6.2.2

Hipothesis: ① $f, f', \dots f^{(n-1)}$ Conts. on $0 \le t \le A$, for any $A > 0$.

② $f, f', \dots, f^{(n-1)}$ of exponential order.

③ $f^{(n)}$ is piecewise cont. on $0 \le t \le A$, Any $A > 0$

Then,

$$\boxed{\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)} \qquad (4.1)$$

Proof. By induction.

Back to our IVP: (1.1), (1.2)

$$\mathcal{L}\{y'' - y' - 6y\} = \mathcal{L}\{0\} = \int_0^\infty e^{-st} 0 \, dt = 0$$

or $\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = 0$

Using previous theorem and Corollaries:

$$s^2 Y(s) - s\, y(0) - y'(0) - sY(s) + y(0) - 6Y(s) = 0$$

or

$$s^2 Y(s) - s - (-1) - sY(s) + 1 - 6Y(s) = 0$$

$$(s^2 - s - 6)Y(s) - s + 2$$

Thus, Solution:

$$\boxed{Y(s) = \frac{s-2}{s^2 - s - 6}} \qquad (4.2)$$

Now, we apply inverse L.T. to recover the Soln. $y(t)$ from $Y(s)$. First, we transform (4.2) into Simple fractions for which L.T are available.

In fact,

$$Y(s) = \frac{s-2}{s^2-s-6} \overset{\text{partial fraction}}{=\!=} \frac{s-2}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$$

$$= \frac{A(s+2)+B(s-3)}{(s-3)(s+2)} = \frac{As+Bs+2A-3B}{(s-3)(s+2)}$$

Therefore,

$$A+B=1, \quad 2A-3B=-2$$

$$\downarrow$$

$$\begin{aligned} -2A-2B &= -2 \\ 2A-3B &= -2 \\ \hline -5B &= -4 \end{aligned} \Rightarrow \boxed{B = 4/5} \Rightarrow \boxed{A = 1-\frac{4}{5} = \frac{1}{5}}$$

Thus,

$$Y(s) = \frac{1}{5}\,\frac{1}{s-3} + \frac{4}{5}\,\frac{1}{s+2}$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$\boxed{y(t) = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}}$$

Solve the IVP

$$\begin{cases} y'' + 2y' + 2y = \cos t & (6.1) \\ y(0) = 1, \quad y'(0) = 0 & (6.2 \end{cases}$$

Using L.T.

$$\mathcal{L}\{y'' + 2y' + 2y\} = \mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \frac{s}{s^2+1}$$

$$s^2 Y(s) - s\overset{=1}{y(0)} - \overset{=0}{y'(0)} + 2sY(s) - 2y(0) + 2Y(s) = \frac{s}{s^2+1}$$

Using ICs:

$$s^2 Y(s) - s + 2sY(s) - 2 + 2Y(s) = \frac{s}{s^2+1}$$

$$\left(s^2 + 2s + 2\right) Y(s) - s - 2 = \frac{s}{s^2+1}$$

Then solving for $Y(s)$:

$$\boxed{Y(s) = \frac{\frac{s}{s^2+1} + s + 2}{s^2 + 2s + 2} = \frac{s^3 + 2s^2 + 2s + 2}{(s^2 + 2s + 2)(s^2+1)}} \qquad (6.3)$$

Hard to get L.T
It needs to be
transformed into
simpler fractions.

Now, using partial fraction decomposition

$$\frac{s^3+2s^2+2s+2}{(s^2+2s+2)(s^2+1)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+1}$$

$$= \frac{(As+B)(s^2+1)+(Cs+D)(s^2+2s+2)}{(s^2+2s+2)(s^2+1)}$$

$$= \frac{As^3+Bs^2+As+B+Cs^3+2Cs^2+2Cs+Ds^2+2Ds+2D}{(s^2+2s+2)\ (s^2+1)}$$

$$= \frac{(A+C)s^3+(B+2C+D)s^2+(A+2C+2D)s+B+2D}{(s^2+2s+2)\ (s^2+1)}$$

System of equations to be solved:

$$\begin{aligned}
s^3&: \quad A \quad\ + C \qquad\quad = 1\\
s^2&: \quad\quad B + 2C + D = 2\\
s^1&: \quad A \quad\ + 2C + 2D = 2\\
1&: \quad\quad B \qquad\ + 2D = 2
\end{aligned}$$

Solving the system

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 1 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 2 & 2 \end{array}\right)\begin{matrix} (-1) \\ \\ \\ \\ \end{matrix} \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 2 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right)\begin{matrix} \\ \to (-1) \\ \\ \\ \end{matrix}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & 1 & 0 \end{array}\right)\begin{matrix} \\ \\ (2) \\ \\ \end{matrix} \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 5 & 2 \end{array}\right)$$

Thus, $5D = 2 \Rightarrow \boxed{D = 2/5}$, $\boxed{C = 1 - 2D = 1 - 4/5 = 1/5}$

$B = 2 - 2C - D = 2 - 2/5 - 2/5 = 6/5$, $\boxed{B = 6/5}$

$\boxed{A = 1 - C = 1 - 1/5 = 4/5}$

Then,

$$\boxed{Y(s) = \frac{1}{5}\,\frac{4s+6}{s^2+2s+2} + \frac{1}{5}\,\frac{s+2}{s^2+1}}$$

Complete square

$$= \frac{1}{5}\left[\frac{4s+6}{(s+1)^2+1} + \frac{s+2}{s^2+1}\right]$$

$$= \frac{1}{5}\left[\frac{4s+4}{(s+1)^2+1} + \frac{2}{(s+1)^2+1} + \frac{s}{s^2+1} + \frac{2}{s^2+1}\right]$$

or

$$\boxed{Y(s) = \frac{4}{5}\,\frac{s+1}{(s+1)^2+1} + \frac{2}{5}\,\frac{1}{(s+1)^2+1} + \frac{1}{5}\,\frac{s}{s^2+1} + \frac{2}{5}\,\frac{1}{s^2+1}}$$

$\downarrow \mathcal{L}^{-1}$ (inverse L.T.)

$$\boxed{y(t) = \frac{4}{5}e^{-t}\cos t + \frac{2}{5}e^{-t}\sin t + \frac{1}{5}\cos t + \frac{2}{5}\sin t}$$