

Laplace Transform

Motivation: Consider the IVP:

$$\begin{cases} y'' - y' - 6y = 0 & (1.1) \\ y(0) = 1, \quad y'(0) = -1 & (1.2) \end{cases}$$

Its soln. can easily be obtained assuming a soln.

Such as $y(x) = e^{rx}$

It leads to a Charact. eqn.

$$r^2 - r - 6 = 0 \Rightarrow r_1 = 3, \quad r_2 = -2$$

$$(r-3)(r+2) = 0$$

$\downarrow$	$\downarrow$	
$y_1 = e^{3t}$	$y_2 = e^{-2t}$	Fund. set.

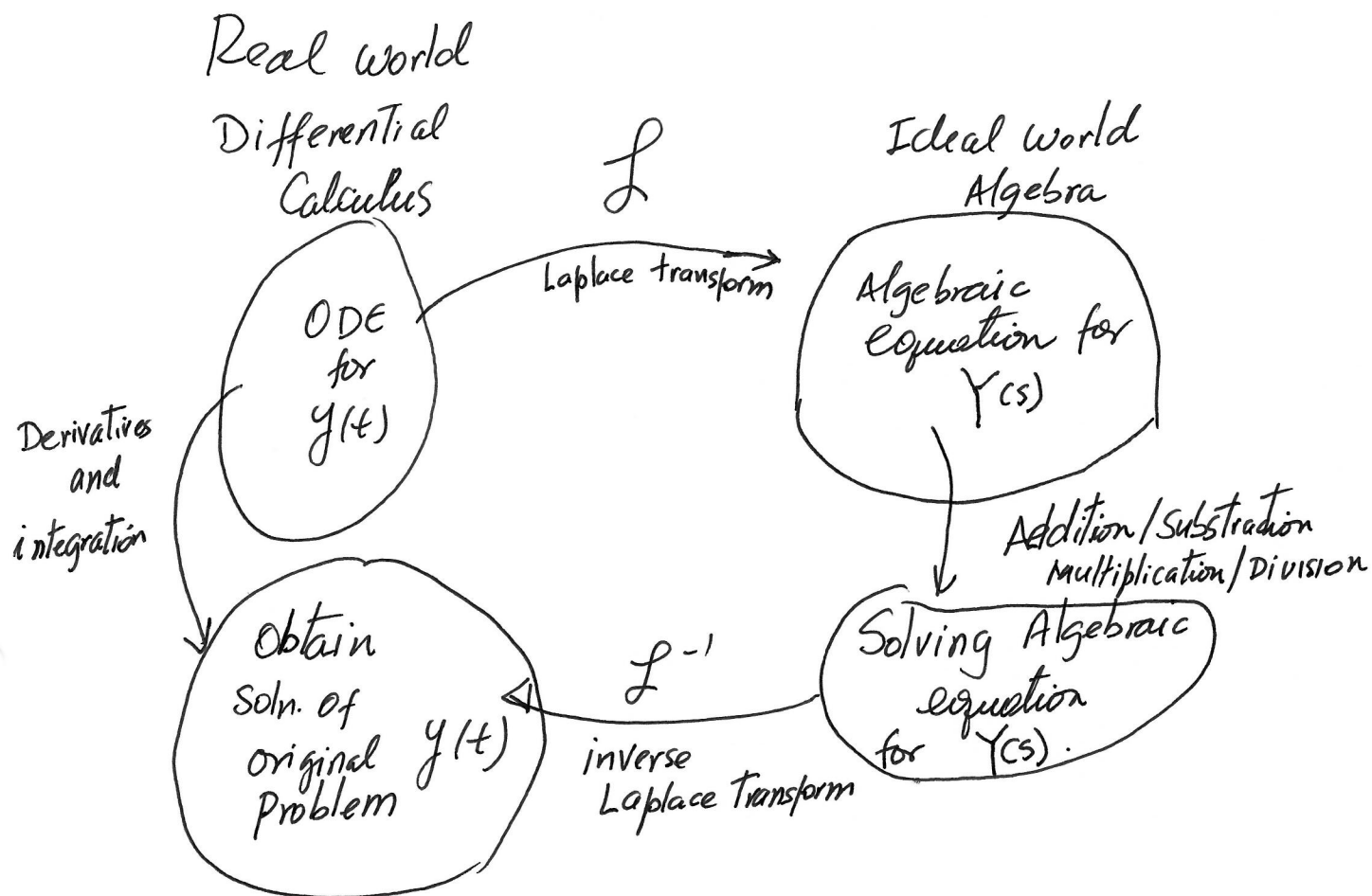
General soln:

$$y(t) = C_1 e^{3t} + C_2 e^{-2t}$$

And using ICs: $y(t) = \frac{1}{5} e^{3t} + \frac{4}{5} e^{-2t} \quad (1.3)$

In the next page, we introduce a completely different technique to solve this IVP. This technique is called "Laplace Transform".

Idea:



$$\mathcal{L} \left\{ \begin{matrix} (1.1) \\ (1.2) \end{matrix} \right\} = (s^2 - s - 6) Y(s) - s + 1 - 1 = \mathcal{L} \{0\} = 0$$

$$\Rightarrow Y(s) = \frac{s-2}{s^2-s-6} \xrightarrow{\text{partial fraction decomp.}} \frac{1}{5} \frac{1}{s-3} + \frac{4}{5} \frac{1}{s+2}$$

$$\xrightarrow{\mathcal{L}^{-1}} y(t) = \frac{1}{5} e^{3t} + \frac{4}{5} e^{-2t}$$

Practical Value:

Consider the nonhomogeneous IVP:

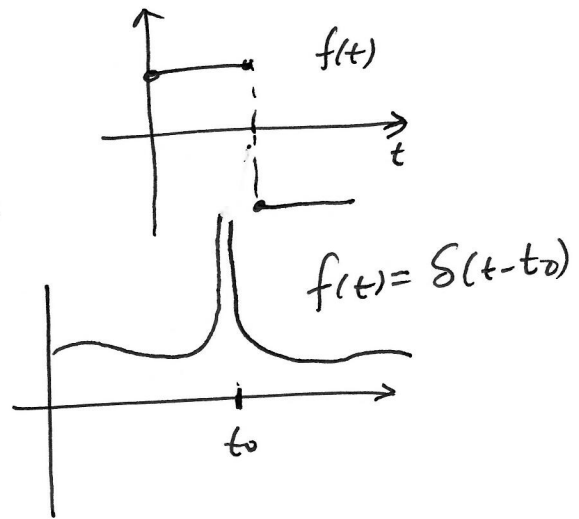
$$\begin{cases} ay'' + by' + cy = f(t) \\ y(0) = y_0, \quad y'(0) = y'_0 \end{cases}$$

Laplace transform very useful if

① $f(t)$ has jump discontinuities

or

② $f(t)$ is an impulse function:



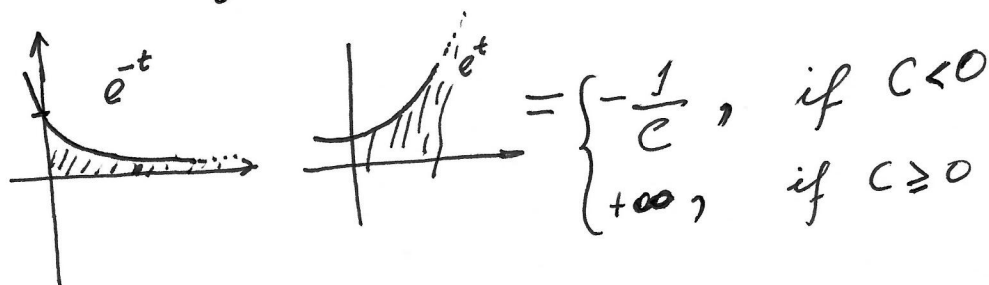
6.1 Definition of Laplace Transform.

Improper Integrals

$$o) \int_0^{\infty} 1 dt = \lim_{A \rightarrow \infty} \int_0^A 1 dt = \lim_{A \rightarrow \infty} t \Big|_0^A = \infty$$

Consider

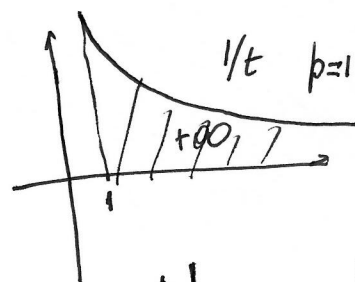
$$i) \int_0^{\infty} e^{ct} dt = \lim_{A \rightarrow \infty} \left. \frac{e^{ct}}{c} \right|_0^A = \lim_{A \rightarrow \infty} \left[\frac{e^{cA} - 1}{c} \right]$$



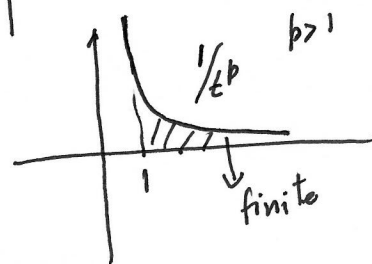
$$ii) \int_1^{\infty} \frac{1}{t} dt = \lim_{A \rightarrow \infty} \ln(t) \Big|_1^A = \lim_{A \rightarrow \infty} [\ln(A) - \ln(1)]$$

$= +\infty$

$$iii) \int_1^{\infty} \frac{1}{t^p} dt = \lim_{A \rightarrow \infty} \int_1^A \frac{1}{t^p} dt =$$



$$= \lim_{A \rightarrow \infty} \int_1^A t^{-p} dt = \lim_{A \rightarrow \infty} \left. \frac{t^{-p+1}}{-p+1} \right|_1^A$$



$$= \lim_{A \rightarrow \infty} \left[\frac{A^{1-p}}{1-p} - \frac{1}{1-p} \right] = \begin{cases} -\frac{1}{1-p} = \frac{1}{p-1}, & p > 1 \\ +\infty, & p \leq 1 \text{ (using ii)} \end{cases}$$

Consider the following improper integrals:

$$i) \int_0^{\infty} e^{-st} \cdot 1 dt = \frac{e^{-st}}{-s} \Big|_0^{\infty} = \frac{e^{-s(\infty)} - e^{-s(0)}}{-s}$$

$s > 0$
fixed

$$= \frac{e^{-\infty} - 1}{-s} = \frac{0 - 1}{-s} = \frac{1}{s}, s > 0$$

So, $\int_0^{\infty} e^{-st} \cdot 1 dt = \frac{1}{s}, s > 0$

$$ii) \int_0^{\infty} e^{-st} \cos(at) dt, \quad s > 0$$

Before solving this integral, let us obtain

$$\int \underbrace{e^{-t}}_f \underbrace{\cos t}_{g'} dt \stackrel{\text{I.P.P.}}{=} \underbrace{e^{-t}}_f \underbrace{\sin t}_g - \int \underbrace{(-1)}_{f'} \underbrace{e^{-t}}_g \sin t dt \quad (2.1)$$

$$\text{Now, } \int \underbrace{e^{-t}}_f \underbrace{\sin t}_{g'} dt \stackrel{\text{I.P.P.}}{=} \underbrace{e^{-t}}_f (-\cos t) - \int \underbrace{(-1)}_{f'} \underbrace{e^{-t}}_g (-\cos t) dt$$

$$= -e^{-t} \cos t - \int e^{-t} \cos t dt \quad (2.2)$$

Substituting (2.2) into (2.1)

$$\int e^{-t} \cos t dt = e^{-t} \sin t - e^{-t} \cos t - \int e^{-t} \cos t dt$$

$$\Rightarrow 2 \int e^{-t} \cos t dt = e^{-t} (\sin t - \cos t)$$

$$\Rightarrow \int e^{-t} \cos t dt = \frac{1}{2} e^{-t} (\sin t - \cos t)$$

$$\begin{aligned}
 i_2) \quad \int_0^{\infty} e^{-st} e^{3t} dt &= \int_0^{\infty} e^{-(s-3)t} dt = \frac{e^{-(s-3)t}}{-(s-3)} \Big|_0^{\infty} \\
 &= \frac{e^{-(s-3)t}}{-(s-3)} \Big|_0^{\infty} = -\frac{1}{(s-3)} [e^{-(s-3)\infty} - e^0] = \frac{1}{s-3}, \quad s > 3.
 \end{aligned}$$

So,

$$\int_0^{\infty} e^{-st} e^{3t} dt = \frac{1}{s-3}, \quad s > 3$$

We will say

$$\mathcal{L}\{e^{3t}\} = \frac{1}{s-3}, \quad s > 3$$

Laplace transform of e^{3t}

In general,

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$$

Recall integration by parts: $\int f g' = fg - \int f g'$

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Now, let's follow a similar procedure to obtain

ii) $\int_0^{\infty} e^{-st} \cos(at) dt = e^{-st} \frac{\sin(at)}{a} \Big|_0^{\infty} - \int_0^{\infty} -s e^{-st} \frac{\sin(at)}{a} dt$, $s > 0$

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$
 f g' f g

Problem
(5) Homework

$$= \frac{e^{-\infty} \sin(a\infty) - e^0 \sin(0)}{a} + \int_0^{\infty} s e^{-st} \frac{\sin(at)}{a} dt$$

$$= \frac{0 - 0}{a} + \frac{s}{a} \int_0^{\infty} e^{-st} \sin(at) dt$$

$$= \frac{s}{a} \int_0^{\infty} e^{-st} \sin(at) dt$$

I.P.P. again

$$= \frac{s}{a} \left[e^{-st} \frac{(-\cos(at))}{a} \Big|_0^{\infty} - \int_0^{\infty} -s e^{-st} \frac{(-\cos(at))}{a} dt \right]$$

$$= \frac{s}{a} \left[\frac{-e^{-\infty} \cos(\infty) - e^0 (-\cos 0)}{a} - \frac{s}{a} \int_0^{\infty} e^{-st} \cos(at) dt \right]$$

$$= \frac{s}{a} \left(\frac{1}{a} \right) - \frac{s^2}{a^2} \int_0^{\infty} e^{-st} \cos(at) dt$$

Then,

$$\int_0^{\infty} e^{-st} \cos(at) dt = \frac{s}{a^2} - \frac{s^2}{a^2} \int_0^{\infty} e^{-st} \cos(at) dt$$

Therefore, if $I = \int_0^{\infty} e^{-st} \cos(at) dt$, then

$$I = \frac{s}{a^2} - \frac{s^2}{a^2} I$$

$$\Rightarrow \left(1 + \frac{s^2}{a^2}\right) I = \frac{s}{a^2}$$

$$\Rightarrow \left(\frac{a^2 + s^2}{a^2}\right) I = \frac{s}{a^2}$$

$$\text{or } I = \frac{s}{a^2 + s^2}, \quad s > 0$$

$$\text{or } \int_0^{\infty} e^{-st} \cos(at) dt = \frac{s}{s^2 + a^2}, \quad s > 0.$$

In particular, if $a = 3$. $\text{or } \mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}, \quad s > 0$

$$\int_0^{\infty} e^{-st} \cos(3t) dt = \frac{s}{s^2 + 3}, \quad s > 0.$$

Similarly,

$$\text{iii) } \int_0^{\infty} e^{-st} \sin(at) dt = \frac{a}{s^2 + a^2}, \quad s > 0$$

Laplace
Transform
 $\sin(at)$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}, \quad s > 0.$$

$$iv) \int_0^{\infty} e^{-st} e^{at} dt, \quad s > a.$$

$$\int_0^{\infty} e^{-st} e^{at} dt = \int_0^{\infty} e^{-(s-a)t} dt$$

$$= \frac{e^{-(s-a)t}}{-(s-a)} \Big|_0^{\infty} = \frac{1}{-(s-a)} [e^{-\infty} - e^0] = \frac{-1}{-(s-a)} = \frac{1}{s-a}$$

or

$$\boxed{\int_0^{\infty} e^{-st} e^{at} dt = \frac{1}{s-a}, \quad s > a}$$

All of these integrals are of the form

$$\boxed{\int_0^{\infty} e^{-st} f(t) dt = F(s)} \quad (5.1)$$

The function $f(t)$ is continuous and

$$|f(t)| \leq \begin{cases} 1, & f(t) = 1 \\ 1, & \cos(at), \sin(at) \\ e^{at}, & f(t) = e^{at} \end{cases}$$

Now, we want to determine what conditions $f(t)$ needs to satisfy for (5.1) to converge

Definition of Laplace Transform

Def. $f(t)$ is piecewise conts on $[\alpha, \beta]$
(p.w.c.)

if there exists a partition of $[\alpha, \beta]$

$$\alpha = t_0 < t_1 < \dots < t_n = \beta$$

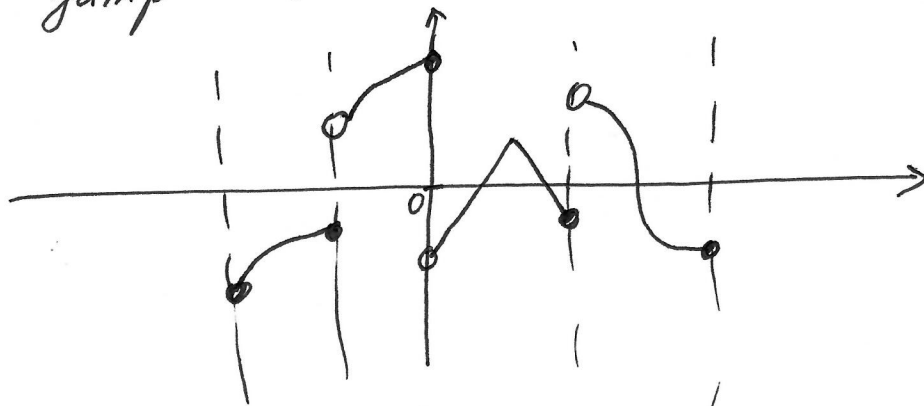
So that

a) f is continuous on each subinterval (t_{i-1}, t_i)
 $i = 1, \dots, n.$

b) f approaches a finite limit as
the endpoints of each subinterval are
approached from within the subinterval

Equivalent to say:

f is p.w.c. on $[\alpha, \beta]$ if it is continuous
in this interval except for a finite number
of jump discontinuities.



Theorem 6.1.1 (Comparison theorem).

- i) $f(t)$ p.w.c., $t \geq a$.
- ii) $|f(t)| \leq g(t)$, $t \geq M > 0$
- iii) $\int_M^\infty g(t) dt$ converges

Then, $\int_a^\infty f(t) dt$ converges

Similarly,

- i) $f(t) \geq g(t) \geq 0$, $t \geq M$
- ii) $\int_0^\infty g(t) dt$ diverges

Then,

$\int_0^\infty f(t) dt$ diverges.

Theorem 6.1.2 If

i) $f(t)$ piecewise continuous (pwc) on $0 \leq t \leq A$
for any $A > 0$.

ii) $|f(t)| \leq K e^{at}$, $t \geq M$, $K, M > 0$ and
 $a \in \mathbb{R}$.

A function $f(t)$ satisfying this condition
is called a "function of exponential order"

Then,

$$I = \int_0^{\infty} e^{-st} f(t) dt \text{ exists for } s > a.$$

(Converge)

Proof.

$$\int_0^{\infty} e^{-st} f(t) dt = \int_0^M e^{-st} f(t) dt + \int_M^{\infty} e^{-st} f(t) dt$$

$$= \underset{\substack{\parallel \\ C_1}}{\text{Constant}} + \int_M^{\infty} e^{-st} f(t) dt$$

Now, from (ii)

$$|e^{-st} f(t)| \leq k e^{-st} e^{at} = k e^{(-s+a)t}, \quad t \geq M$$

Then,

$$\int_0^{\infty} k e^{(a-s)t} dt = \lim_{A \rightarrow \infty} \int_M^A k e^{(a-s)t} dt$$

$$= k \lim_{A \rightarrow \infty} \left[\frac{e^{(a-s)t}}{a-s} \Big|_M^A \right]$$

$$= \frac{k}{a-s} \lim_{A \rightarrow \infty} \left[e^{(a-s)A} - e^{(a-s)M} \right]$$

$$\text{If } s > a \\ = k \frac{e^{(a-s)M}}{s-a} = C_2, \quad s > a$$

Therefore, $\int_0^{\infty} k e^{(a-s)t} = C_1 + C_2$ Converges

Comparison theorem implies

$\int_0^{\infty} e^{-st} f(t) dt$ also converges.

By comparison theorem

$$\int_0^{\infty} e^{-st} f(t) dt \text{ converges.}$$

Definition (Laplace transform)

If $f(t)$ satisfies conditions (i) and (ii) of thm 6.1.2 :

- i) $f(t)$ is p.w.c. on $0 \leq t \leq A$, for any $A > 0$.
- ii) $f(t)$ is of exponential order.

Then,

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt, \quad s > a$$

is called the Laplace transform of $f(t)$.

The Laplace transform can be seen as an integral operator :

$$\mathcal{L} : \left\{ \begin{array}{l} \text{p.w.c.} \\ \text{of exp order} \end{array} \right\} \longrightarrow \begin{array}{l} \text{Set of} \\ \text{functions} \end{array}$$

$$f(t) \longrightarrow \mathcal{L}\{f(t)\} = F(s).$$

Linearity of $\mathcal{L}$

Assume $f_1(t)$ and $f_2(t)$ have Laplace transforms for $s > a_1$ and $s > a_2$, respectively

then for $s > \max(a_1, a_2)$

$$\mathcal{L}\{C_1 f_1(t) + C_2 f_2(t)\}$$

$$= \int_0^{\infty} (C_1 f_1(t) + C_2 f_2(t)) e^{-st} dt$$

$$= C_1 \int_0^{\infty} e^{-st} f_1(t) dt + C_2 \int_0^{\infty} e^{-st} f_2(t) dt$$

$$= C_1 \mathcal{L}\{f_1(t)\} + C_2 \mathcal{L}\{f_2(t)\}.$$

Therefore, the Laplace transform is a linear operator.

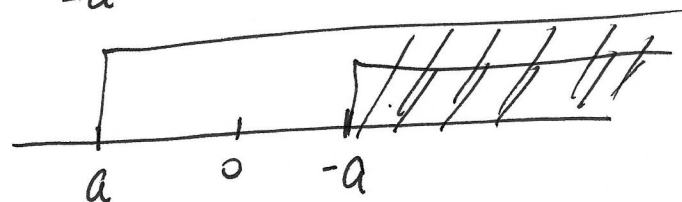
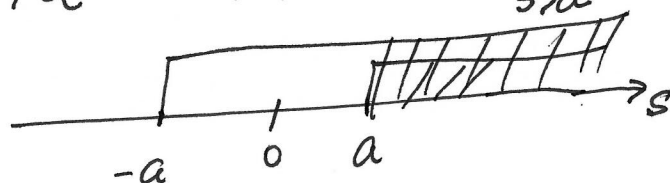
Application:

$$v) \mathcal{L}\{\cosh(at)\} = \mathcal{L}\left\{\frac{e^{at} + e^{-at}}{2}\right\}$$

linearity

$$= \frac{1}{2} \mathcal{L}\{e^{at}\} + \frac{1}{2} \mathcal{L}\{e^{-at}\}$$

$$= \frac{1}{2} \frac{1}{s-a} + \frac{1}{2} \frac{1}{s+a}, \quad s > a, \text{ if } a > 0, \quad s > -a, \quad \downarrow \quad s > a$$



If $a < 0$, $s > a \Rightarrow s > -a$
 $s > -a$

Both cases $s > |a|$.

$$= \frac{\frac{1}{2}(s+a) + \frac{1}{2}(s-a)}{(s-a)(s+a)} = \frac{s}{s^2 - a^2};$$

Therefore,

$$\boxed{\mathcal{L}\{\cosh(at)\} = \frac{s}{s^2 - a^2}, \quad s > |a|}$$

Similarly,

$$\begin{aligned}
 \text{VI) } \mathcal{L}\{\sinh(at)\} &= \frac{1}{2} \mathcal{L}\{e^{at}\} - \frac{1}{2} \mathcal{L}\{e^{-at}\} \\
 &= \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right] = \frac{1}{2} \left(\frac{s+a-s+a}{s^2-a^2} \right) \\
 &= \frac{a}{s^2-a^2}, \quad s > |a|
 \end{aligned}$$

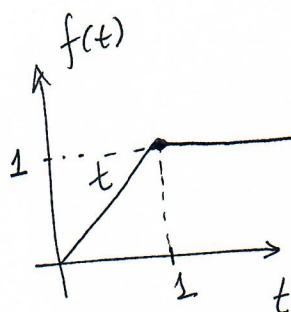
Then,

$$\boxed{\mathcal{L}\{\sinh(at)\} = \frac{a}{s^2-a^2}, \quad s > |a|.}$$

6.12

#17)
Section 6.1
(11th Ed.)

$$f(t) = \begin{cases} t, & 0 < t < 1 \\ 1, & 1 \leq t < \infty \end{cases}$$



$$\Rightarrow \int_0^1 t e^{-st} dt + \int_1^{\infty} e^{-st} dt$$

$$\stackrel{s > 0}{=} \left. t \frac{e^{-st}}{-s} \right|_0^1 - \int_0^1 \frac{e^{-st}}{-s} dt + \left. \frac{e^{-st}}{-s} \right|_1^{\infty}$$

$$= -\frac{e^{-s}}{s} + \frac{e^{-st}}{-s^2} \Big|_0^1 + \frac{e^{-s}}{s} = -\frac{e^{-s}}{s} + \frac{e^{-s}}{-s^2} + \frac{1}{s^2} + \frac{e^{-s}}{s}$$

$$\frac{e^{-s}}{s^2} + \frac{1}{s^2} = \frac{(1-e^{-s})}{s^2}, \quad s > 0.$$