5.1 <u>Review of Power Series.</u> $0^{\circ\prime}$

Def. – A power series is an infinite series of the form:

$$S(x) = \sum_{n=1}^{\infty} a_n (x-x_0)^n \qquad (1)$$

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(1) always converge for $x = x_0$

$$S(x_0) = 0.$$

Def. – $S(x)$ is abs. convergent if

$$\sum_{n=1}^{\infty} \left| a_n (x-x_0)^n \right| = \sum_{n=1}^{\infty} |a_n| \, |x-x_0|^n \quad \text{converges.}$$

Def. – There is a nonnegative number ρ, called radius of convergence, such that

$$\sum_{n=1}^{\infty} a_n (x-x_0)^n \quad \text{converges absolutely for } |x-x_0| < \rho$$

and diverges for $|x-x_0| > \rho$

Rmks: – If $S(x)$ only converges at $x = x_0$, then $\rho = 0$

– At x such that $|x-x_0| = \rho$, the series may or may not converge.

Thm If the power series

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

is absolutely convergent, then it is convergent.

The converse is not true. It means the infinite series may converge but its absolute value not.

Example: $$S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$ converges (alternating series test)

However, $$S = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3}$$ diverges.

This is called the harmonic series.

<u>Ratio Test</u> Consider $\sum_{n=0}^{\infty} a_n (x-x_0)^n$

If $a_n \neq 0$ and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$,
then for a fixed value of x

$$\lim_{n\to\infty} \left| \frac{a_{n+1} (x-x_0)^{n+1}}{a_n (x-x_0)^n} \right| = \lim_{n\to\infty} |x-x_0| \left|\frac{a_{n+1}}{a_n}\right|$$

$$= |x-x_0| L$$

Therefore at that value of x,

1) the series converges absolutely if $|x-x_0| L < 1$
2) " " diverges if $|x-x_0| L > 1$.
3) The test is inconclusive if $|x-x_0| L = 1$.

Example 2. (book)

Determine radius of convergence of

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n\,2^n}$$

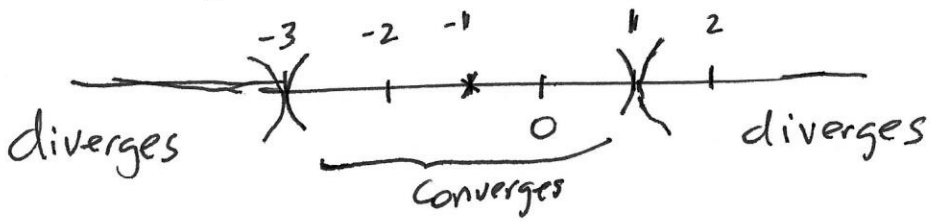
Applying ratio test:

$$\lim_{n\to\infty} \left| \frac{\frac{(x+1)^{n+1}}{(n+1)2^{n+1}}}{\frac{(x+1)^n}{n\,2^n}} \right| = \lim_{n\to\infty} \left| \frac{(x+1)\,n}{2(n+1)} \right|$$

$$= \frac{|x+1|}{2} \lim_{n\to\infty} \frac{n}{n+1} = \frac{|x+1|}{2}$$

So the P. Series converges if

$$\frac{|x+1|}{2} < 1 \Rightarrow |x+1| < 2 \Rightarrow -3 < x < 1$$



at $\boxed{x = -3}$

$$S(-3) = \sum_{n=1}^{\infty} \frac{(-3+1)^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{(-2)^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{-2}{2}\right)^n$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \text{ Converges.}$$

However, at $\boxed{x=1}$

$$S(1) = \sum_{n=1}^{\infty} \frac{2^n}{n 2^n} = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

So interval of convergence is given by

$[-3, 1)$. and $\rho = 2$.

If $\sum_{n=0}^{\infty} a_n (x-x_0)^n \rightarrow f(x)$.

or $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ and $\underbrace{|x-x_0| < \rho}_{\text{(interval of convergence)}}$, $\rho > 0$.

then f is conts. and has derivatives of all orders.
for $|x-x_0| < \rho$.

Moreover,

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$$

$$f''(x) = \sum n(n-1) a_n (x-x_0)^{n-2}$$

$\vdots$

differentiation term by term is valid.