High order

4.2 Homogeneous ODEs. with constant coeffs.

$$ay'' + by' + cy = 0 \qquad 2^{nd} \text{ order}$$

Examples of high order:

i) $y'''' - y''' - y'' + y' = 0$

or $y^{(4)} - y^{(3)} - y^{(2)} + y' = 0$

ii) $y'''''' + y''' = y^{(6)} + y^{(3)} = 0$

(i) and (ii) are particular cases of

$$\boxed{a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0} \qquad (1.1)$$

We want to obtain the general soln. of (1.1)

From here, we imitate what we did for 2^{nd} order in sections 3.2–3.4.

By assuming a solution of (1.1) as $y(t) = e^{rt}$ we arrive to a characteristic polynomial

$$\cancel{e^{rt}}\left(a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n\right) = 0 \qquad (1.2)$$

For this n^{th} order polynomial, there are many possibilities for the nature of its roots.

i) Real and different
ii) Real and some repeated.
iv) Real, different and Complex
v) Real some repeated and other complex and they also can be repeated, etc.

Let us analyze the Char. polyn. of (i) and (ii)

For (i), $\rightarrow$ Real roots

Char. polyn: $\boxed{r^4 - r^3 - r^2 + r = 0}$ (2.1)

$r(r^3 - r^2 - r + 1) = 0 \Rightarrow \underline{r_1 = 0}$

Recall the rational root theorem.

"If $r = p/q$ is a rational root, (p, q w/o common factors) then p should be a factor of a_n, and q must be a factor of a_0."

$r^3 - r^2 - r + 1 = 0$

In our case, $p = 1$, $q = 1$, then a possible root is $r_2 = \frac{1}{1} = 1$.

So, let us substitute $r_2 = 1$ into

$$r^3 - r^2 - r + 1 = 0$$

$$\cancel{1^3} - \cancel{1^2} - \cancel{1} + \cancel{1} = 0 \checkmark$$

So $r_2 = 1$ is indeed a soln. of $\underline{r^3 - r^2 - r + 1 = 0}$

and we can represent (2.1) as

$$r^4 - r^3 - r^2 + r = r(r-1)\,P_2(r) \overset{?}{=} 0$$

To obtain $P_2(r)$ use long division

$$\begin{array}{l|l} \cancel{r^3} - \cancel{r^2} - r + 1 & r - 1 \\ \hline -\cancel{r^3} + \cancel{r^2} & r^2 - 1 \\ \hline -r + 1 & \\ +r - 1 & \\ \hline 0\;0 & \end{array}$$

Then, $r^4 - r^3 - r^2 + r = r(r-1)(r^2-1)$

$$= r(r-1)(r-1)(r+1) = r(r-1)^2(r+1)$$

$y_1(r) \equiv 1$, $y_2(r) = e^t$, $y_3(r) = te^t$, $y_4(t) = e^{-t}$

Therefore,

(3.1) $$\boxed{y(t) = C_1 + C_2 e^t + C_3 t e^t + C_4 e^{-t}.}$$

It can be shown that $W(y_1, y_2, y_3, y_4) \neq 0$ for all t.

Thus, (3.1) is the General Soln. of (i).

4.2 Higher Order homogeneous equations With Constant Coefficients. (Complex roots)

Find the general Soln. of

$$y^{(4)} - y = 0$$

Answ: $r^4 - 1 = 0 \Rightarrow r^4 = 1 \Rightarrow r = (1)^{1/4}$

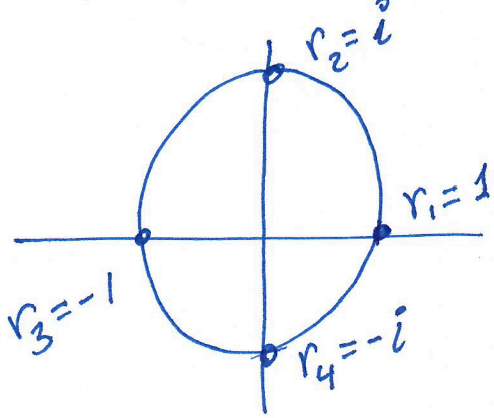
Clearly two roots are

$r_1 = 1, \quad r_2 = -1$

two other less obvious roots are

$r_3 = i, \quad r_4 = -i.$ since $(\pm i)^4 = (\pm i)^2 (\pm i)^2 = (-1)(-1) = 1$

If we take a look to the unit circle.



So they are obtained by dividing the unit Circle in four equal pieces, starting at the x-axis.

They Can also be obtained more Systematically as

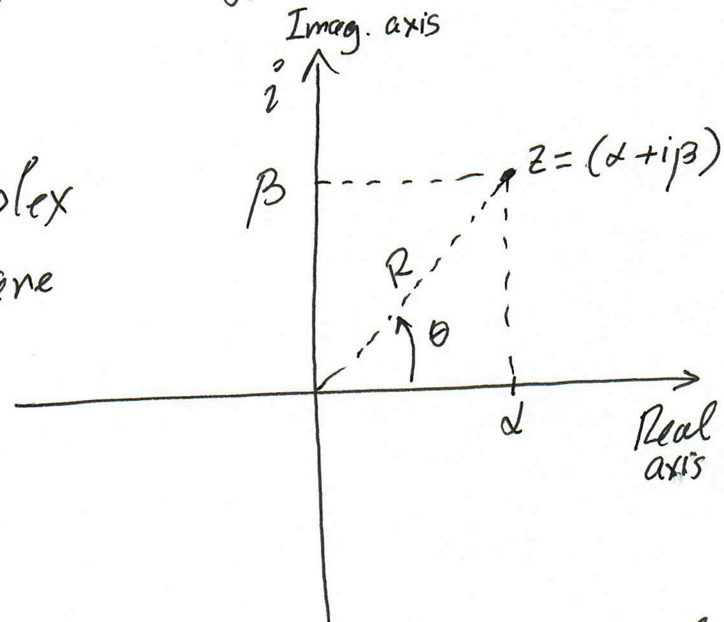
$$r_m = (1)^{1/4} = \left[e^{i(0 + 2m\pi)}\right]^{1/4} = e^{i\frac{2m\pi}{4}}$$

$$= e^{i\frac{m\pi}{2}} = \begin{cases} e^{i0} = 1, & m=0 \\ e^{i\pi/2} = \cos \pi/2 + i \sin \pi/2 = i, & m=1 \\ e^{i4\pi/4} \cos \pi + i \sin \pi = -1, & m=2 \\ e^{i6\pi/4} = e^{i3\pi/2} = \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} = -i, & m=3 \end{cases}$$

Complex Numbers and the plane $\mathbb{R}^2$

A complex number $z = \alpha + i\beta$ can be represented in the plane $\mathbb{R}^2$, by assigning a pair (x, y) such that $x = \alpha$ and $y = \beta$

Complex plane



$R = \sqrt{\alpha^2 + \beta^2}$

$\tan\theta = \frac{\beta}{\alpha}$

Therefore, the complex number z also has a polar representation given by

$$\alpha = R\cos\theta, \quad \beta = R\sin\theta$$

$$\boxed{\alpha + i\beta = R\cos\theta + iR\sin\theta = R(\cos\theta + i\sin\theta) = Re^{i\theta} = Re^{i(\theta + 2m\pi)} \quad m = 0, 1, 2 \ldots}$$

where $\boxed{R = \sqrt{\alpha^2 + \beta^2}}$

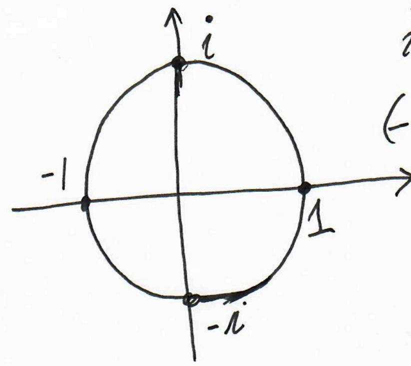
and
$$\boxed{\theta = \begin{cases} \arctan\left(\frac{\beta}{\alpha}\right), & 1^{st} \text{ quad} \\ \arctan(\beta/\alpha) + \pi, & 2^{nd} \text{ and } 3^{rd} \text{ quad.} \\ \arctan(\beta/\alpha) + 2\pi, & 4th \text{ quad.} \end{cases}}$$

Consider

$\sqrt{1} \stackrel{?}{=}$, $\sqrt{1} = \pm 1$ because $(+1)^2 = 1$
and $(-1)^2 = 1$

Now, consider

$$\sqrt[4]{1} = \begin{cases} 1 \\ -1 \\ ?\ \ i \\ ?\ -i \end{cases}$$



$i^4 \stackrel{?}{=} i^2 \cdot i^2 = (-1)(-1) = 1$

$(-i)^4 = (-i)^2(-i)^2 =$
$= i^2 \cdot i^2 = 1$

Systematic way:

$1 \stackrel{?}{=} Re^{i\theta}$, $R \pm \sqrt{(1)^2 + (0)^2} = +\sqrt{1} = 1$

$1 + 0i$
$\alpha + i\beta$

$\tan\theta = \frac{\beta}{\alpha} = \frac{0}{\alpha} = 0 \Rightarrow \theta = \arctan(0) = 0$

and $\boxed{1 = Re^{i\theta} = 1\, e^{i0} = 1\, e^{i(0+2m\pi)}}$

Then, $\sqrt[4]{1} = \left[e^{i(0+2m\pi)}\right]^{1/4} = e^{i\frac{2m\pi}{4}} = \boxed{e^{im\pi/2}, \quad m = 0, 1, 2, \ldots}$

$m=0: \quad e^{i0} = 1$

$m=1: \quad e^{i\pi/2} = i$

$m=2: \quad e^{i\pi} = -1$

$m=3: \quad e^{i\frac{3\pi}{2}} = -i$

$m=4: \quad e^{i2\pi} = 1$

$m=5: \quad e^{i5\pi/2} = i$

$m=6: \quad e^{i6\pi/2} = e^{i3\pi} = -1$

$m=7: \quad e^{i7\pi/2} = -i$

So, we only need to compute roots for $m = 0, 1, 2, 3$

Therefore, the roots for

$$\sqrt[4]{1}$$

are given by

	Solutions of ODE
$r_1 = 1$	$\Rightarrow \; y_1(t) = e^t$
$r_2 = i$	$\Rightarrow \; y_2(t) = \cos t$
$r_3 = -1$	$\Rightarrow \; y_3(t) = e^{-t}$
$r_4 = -i$	$\Rightarrow \; y_4(t) = \sin t$

The general soln. is given by

$$\boxed{y(t) = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t.}$$

A faster alternative to obtain the soln. of $y^{(4)} - y = 0$ is to factor the charact. equ. such as,

$$r^4 - 1 = 0 \Rightarrow (r^2 - 1)(r^2 + 1) = 0$$

$$r^2 = 1 \Rightarrow r = \pm 1, \qquad r^2 = -1 \Rightarrow r = \pm\sqrt{-1} = \pm i$$

So, $y(t) = C_1 e^t + C_2 e^{-t} + C_3 \cos(t) + i \sin(t).$

For (ii) (less obvious example)

<u>Complex roots:</u>

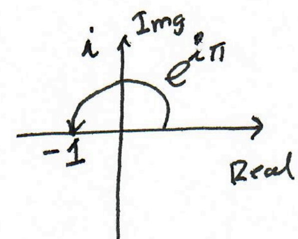
$$y^{(6)} + y^{(3)} = 0$$

Char. polyn: $r^6 + r^3 = 0 \Rightarrow r^3(r^3+1) = 0$

From here, $r_1 = 0$ is a root, then

$$\boxed{y_1(t) = e^{0t} = 1} \text{ is a soln. } (8.1)$$

the other roots and corresponding solns. can be obtained from $\boxed{r^3+1=0}$



$$r^3 = -1 \Rightarrow r = (-1)^{1/3}$$

Now, $-1 = e^{i\pi} = \cos\pi + i\sin\pi$
$(-1 + i * 0 = -1)$.

But also, $(-1) = e^{i(\pi + 2m\pi)}$, $m = 0, 1, 2 \ldots$

Then,

$$\boxed{\begin{aligned} (-1)^{1/3} &= \left(e^{i(\pi+2m\pi)}\right)^{1/3} \\ &= e^{i(\pi/3 + \frac{2}{3}m\pi)}, \quad m = 0, 1, 2 \ldots \end{aligned}} \quad (8.2)$$

Now, we can obtain all the roots of

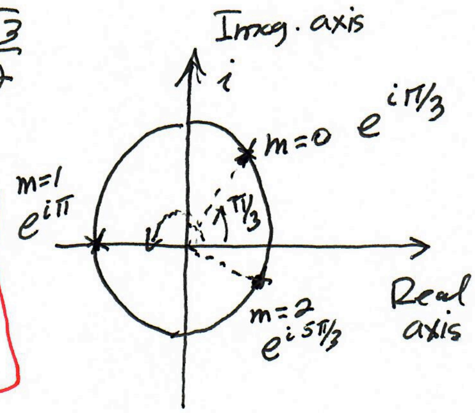
$$r^3 + 1 = 0$$

by using formula (8.2) for $m = 0, 1$.

In fact,

$$\underline{\underline{m=0}} \Rightarrow e^{i(\pi/3 + 0)} = \cos(\pi/3) + i \sin(\pi/3)$$

$$= \tfrac{1}{2} + i \frac{\sqrt{3}}{2}$$



Thus, we have two solutions:

$$\boxed{y_1(t) = e^{t/2} \cos\left(\frac{\sqrt{3}}{2} t\right)}$$

$$\boxed{y_2(t) = e^{t/2} \sin\left(\sqrt{3}/2\, t\right)}$$

$$\underline{\underline{m=1}} \Rightarrow e^{i(\pi/3 + 2\pi/3)} = e^{i\pi} = -1$$

So, $\boxed{y_3(t) = e^{-t}}$

$$\underline{\underline{m=2}} \Rightarrow e^{i(\pi/3 + 4\pi/3)} = e^{i(5\pi/3)}$$

$$= \cos(5\pi/3) + i \sin(5\pi/3) = \frac{1}{2} - \frac{\sqrt{3}}{2} i$$

From here, we have two more solutions:

$y_4(t) = e^{t/2} \cos\left(-\frac{\sqrt{3}}{2} t\right)$ Same as $y_1(t)$

$y_5(t) = e^{t/2} \sin\left(-\frac{\sqrt{3}}{2} t\right) = - e^{t/2} \sin\left(\frac{\sqrt{3}}{2} t\right)$

multiple of $y_2(t)$.

For larger $m = 3, 4, \ldots$, we obtain the Same roots (peridiocity) that we already have.

So, the three roots for $r^3 + 1 = 0$ and the Corresponding Solutions: y_1, y_2 and y_3 are:

$m = 0$, $r_1 = \frac{1}{2} + i\frac{\sqrt{3}}{2} \longrightarrow$ $y_1(t) = e^{t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)$

$m = 1$, $r_2 = -1 \longrightarrow y_2(t) = e^{-t}$

$m = 2$, $r_3 = \frac{1}{2} - i\frac{\sqrt{3}}{2} \longrightarrow$ $y_3(t) = e^{t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$

Summarizing,

$$\boxed{y^{(6)} + y^{(3)} = 0}$$

has a Charact. polyn: $r^6 + r^3 = 0 \Leftrightarrow r^3(r^3+1) = 0$

roots are $r_1 = 0$ (repeated three times)

$r_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, $r_3 = -1$, $r_4 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

The solutions constructed from these roots are

$y_1(t) = 1$, $y_2(t) = t$, $y_3(t) = t^2$ from $r_1 = 0$.

$y_4(t) = e^{-t}$ from $r_3 = -1$

$y_5(t) = e^{\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)$, $y_6(t) = e^{\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)$ from r_2 and r_4

and the general solution of

$$y^{(6)} + y^{(3)} = 0$$

is given by

$$\boxed{y(t) = C_1 + C_2 t + C_3 t^2 + C_4 e^{t/2} \cos\left(\frac{\sqrt{3}}{2}t\right) + C_5 e^{t/2} \sin\left(\frac{\sqrt{3}}{2}t\right) + C_6 e^{-t}}$$

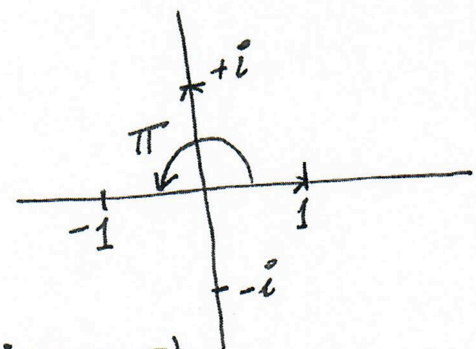
4.2 Homogeneous Equations with Constant Coefficients

Exercise #11 Find the general solution of

$$y^{(6)} + y = 0$$

Answ: $r^6 + 1 = 0 \Rightarrow r^6 = -1 \Rightarrow r = (-1)^{1/6}$

Complex plane:



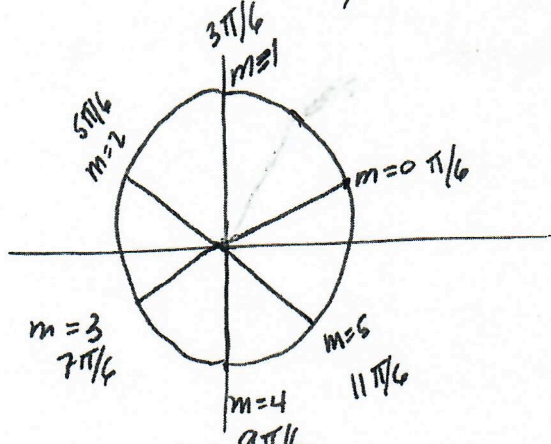
$-1 = e^{i\pi} \Rightarrow$

Then $(-1)^{1/6} = \left(e^{i\pi}\right)^{1/6}$
$= e^{i\pi/6}$

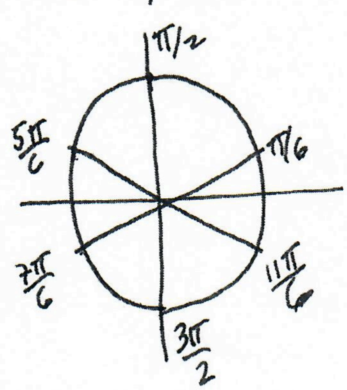
but also, $-1 = e^{i(\pi + 2m\pi)}$, $m = 0, 1, 2 \ldots$ Counterclockwise

Therefore, $(-1)^{1/6} = \left[e^{i(\pi + 2m\pi)}\right]^{1/6} = e^{i\left(\pi/6 + 2m\frac{\pi}{6}\right)}$

$= e^{i(\pi/6 + m\pi/3)}$, $m = 0, 1, .., 5$



More precise



These roots can be written in terms of trig. functions and from these representation we can obtain the corresponding solutions:

$m=0$ $\quad e^{i\pi/6} = \cos(\pi/6) + i\sin(\pi/6)$

$$= \frac{\sqrt{3}}{2} + i\,\frac{1}{2}$$

$$y_1(t) = e^{\sqrt{3}/2\,t}\cos(t/2), \quad y_2(t) = e^{\sqrt{3}/2\,t}\sin(t/2).$$

$m=1$ $\quad e^{i\pi/2} = \cos(\overset{=0}{\pi/2}) + i\sin(\overset{=1}{\pi/2}) = i$

$$y_3(t) = \sin t, \quad y_4(t) = \cos t$$

$m=2$ $\quad e^{i\frac{5\pi}{6}} = \cos\left(\frac{5\pi}{6}\right) + i\sin(5\pi/6) = -\frac{\sqrt{3}}{2} + i\,\frac{1}{2}$

$$y_5(t) = e^{-\sqrt{3}/2\,t}\cos(t/2), \quad y_6(t) = e^{-\frac{\sqrt{3}}{2}t}\sin(t/2)$$

$m=3$ $\quad e^{i7\pi/6} = \cos(7\pi/6) + i\sin(7\pi/6) = -\frac{\sqrt{3}}{2} - i\,\frac{1}{2}$

Complex conjugate to $m=2$

$$y_7(t) = e^{-\sqrt{3}/2\,t}\cos(-t/2) = y_5(t),$$

$$y_8(t) = e^{-\sqrt{3}/2\,t}\sin(-t/2) = -e^{-\sqrt{3}/2\,t}\sin(t/2) = -y_6(t)$$

$m=4$ $\quad e^{i\frac{3\pi}{2}} = \cos\overset{=0}{(3\pi/2)} + i\sin\overset{=1}{(3\pi/2)} = -1i = -i$

Complex conjugate to the case $m=1$

Same solns: $y_3(t) = \sin t$, $\quad y_4(t) = \cos t$.

$m=5$ $\quad e^{i\,11\pi/6} = \cos\left(\frac{11\pi}{6}\right) + i\sin(11\pi/6)$

$$= \frac{\sqrt{3}}{2} - i\frac{1}{2}$$

Complex conjugate of roots $m=0$ case.

Same solns:

$$y_1(t) = e^{\sqrt{3}/2\,t}\cos(t/2), \quad y_2(t) = e^{\sqrt{3}/2\,t}\sin(t/2).$$

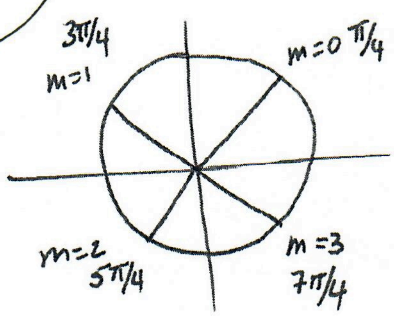
So general soln:

$$\boxed{\begin{aligned} y(t) = {} & C_1 e^{\sqrt{3}/2\,t}\cos(t/2) + C_2 e^{\sqrt{3}/2\,t}\sin(t/2) \\ & + C_3\sin t + C_4\cos t + C_5 e^{-\sqrt{3}/2\,t}\cos(t/2) + C_6 e^{-\sqrt{3}/2\,t}\sin(t/2) \end{aligned}} \qquad (H3.1)$$

Try: Omit this $\quad y^{(4)} + 4y = 0 \qquad r^4 + 4 = 0 \Rightarrow r^4 = \overset{=4(-1)}{-4} = (4e^{i\pi})$

or $\quad r = (4e^{i\pi})^{1/4} = (4)^{1/4} e^{i\left(\frac{\pi}{4} + \frac{2m\pi}{4}\right)}$

or $\quad r = 4^{1/4} e^{i(\pi/4 + m\pi/2)}$, $\quad m = 0, 1, 2, 3$



Exercise 22

$$y^{(4)} + 2y'' + y = 0$$

$$r^4 + 2r^2 + 1 = 0$$

possible roots : $r = \pm 1$, but none of them is a root.

Try factorization:

$$0 = r^4 + 2r^2 + 1 = (r^2 + 1)^2$$

Then the roots of $r^2 = -1 \Rightarrow r = \pm i$

are repeated twice:

and $\boxed{y(t) = C_1 \cos t + C_2 \sin t + C_3 t \cos t + C_4 t \sin t}$

(H4.1)