Summarizing example 1 in book

Spring-mass system mathematical model.

$$m\,u''(t) + \gamma\,u'(t) + K\,u(t) = F_{ext}(t) \qquad (S.1)$$

Weight $= mg = 4\,lb \Rightarrow \boxed{m = \frac{4}{32} = \frac{1}{8}}$

displacement from natural length ℓ : $\boxed{L = 2in = \frac{1}{6}\,ft.}$

Drag force : $F_d = 6\,lb$ when $u'(t) = 3\,ft/s$

then $\gamma \overset{=3}{u'} = 6$ or $\boxed{\gamma = 2}$

Substituting into (S.1)

$$\frac{1}{8}u'' + 2u' + Ku = 0 \qquad (S.2)$$

Now, K can be computed from the equilibrium relationship

$$mg = KL \quad (\text{Spring-mass system at rest})$$
$$(u \equiv 0,\ u' \equiv 0,\ u'' \equiv 0)$$

Then,

$$K = \frac{mg}{L} = \frac{4}{1/6} = 24$$

Finally, IVP to be solved is given by

$$\text{IVP} \quad \begin{cases} u'' + 16u' + 192u = 0 \\ u(0) = 6in = \frac{1}{2}\,ft.\,, \quad u'(0) = 0 \end{cases} \qquad (S.3)$$

<u>Charact. polynomial:</u>

$$r^2 + 16r + 192 = 0$$

$$r_{1,2} = -8 \pm 8\sqrt{2}\, i$$

Then, $$u(t) = A\, e^{-8t} \cos(8\sqrt{2}t) + B\, e^{-8t} \sin(8\sqrt{2}t)$$

Using ICs.

$$\boxed{u(t) = \frac{1}{2}\, e^{-8t} \cos(8\sqrt{2}t) + \frac{1}{4}\sqrt{2}\, e^{-8t} \sin(8\sqrt{2}t).} \quad (S.4).$$

Show graph in MAPLE.

3.7 Damped Free Vibration. No external Force.

$$\boxed{m u'' + \gamma u' + k u = 0} \qquad (S.5)$$

$, \ m, \gamma, k > 0$

Charact. equ.: $m r^2 + \gamma r + k = 0$

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

If (I) $\gamma^2 - 4mk > 0 \Rightarrow u = A e^{\overset{<0}{r_1} t} + B e^{\overset{<0}{r_2} t}$

(II) $\gamma^2 - 4mk = 0 \Rightarrow u = A e^{\overset{<0}{r_1} t} + B t e^{\overset{<0}{r_1} t}$

(III) $\gamma^2 - 4mk < 0 \Rightarrow u = A e^{-\frac{\gamma}{2m} t} \cos(\ \ t) + B e^{-\frac{\gamma}{2m} t} \sin(\ \ t)$

$$\frac{\sqrt{4mk - \gamma^2}}{2m}$$

$$r_1 = -\frac{\gamma}{2m} + \frac{\sqrt{4mk - \gamma^2}}{2m} i$$

$$r_2 = -\frac{\gamma}{2m} - \frac{\sqrt{4mk - \gamma^2}}{2m} i$$

To study the effects of damping, let's consider $m = 1$, $k = 1$, and $\gamma = c$ variable. Then, we arrive to

$$u'' + c u' + u = 0$$

Show solns. depending on c in MAPLE!

Specific Solns. for Damped Vibrations without external force

$$mu'' + cu' + ku = 0, \quad m, c, k > 0$$

If $m=1$, $k=1$, then

$$\text{IVP} \quad \begin{cases} u'' + cu' + u = 0 \\ u(0) = 1, \; u'(0) = 1 \end{cases}$$

General soln:

$$u(t) = c_1 e^{\left(\frac{-c+\sqrt{c^2-4}}{2}\right)t} + c_2 e^{\left(\frac{-c-\sqrt{c^2-4}}{2}\right)t}$$

Overdamped: $c=3$

$$u(t) = \left(\frac{1+\sqrt{5}}{2}\right) e^{\frac{-3+\sqrt{5}}{2}t} + \left(\frac{1-\sqrt{5}}{2}\right) e^{\left(\frac{-3-\sqrt{5}}{2}\right)t}$$

Critically damped: $c=2$

$$u(t) = e^{-t} + 2te^{-t}$$

Damped: $c = 1/10$

$$u(t) = \frac{\sqrt{399}}{19} e^{-\frac{t}{20}} \sin\left(\frac{\sqrt{399}}{20}t\right) + e^{-\frac{t}{20}} \cos\left(\frac{\sqrt{399}}{20}t\right)$$

Undamped Free Vibrations.

In this case $\underline{\gamma = 0}$ (ideal case because there is always damping)

Equation (S.5) reduces to

$$m u'' + K u = 0 \quad \text{or} \quad u'' + \frac{K}{m} u = 0$$

Charact. polyn.: $r^2 + \frac{K}{m} = 0 \Rightarrow r = \pm i \left(\frac{K}{m}\right)^{1/2}$

and general Soln.:

$$u(t) = A \cos\left(\sqrt{\tfrac{K}{m}}\, t\right) + B \sin\left(\sqrt{\tfrac{K}{m}}\, t\right) \qquad (S.6)$$

Usually a new constant

$$\omega_0^2 = K/m \Rightarrow \boxed{\omega_0 = \sqrt{\tfrac{K}{m}}}$$

which is called natural frequency of vibration is introduced and (S.6) is written as

$$\boxed{u(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t)} \qquad (S.7)$$

A more useful representation is discussed in the next page.

$$mu'' + Ku = 0$$

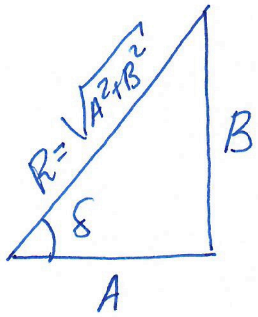
$$r^2 + \frac{K}{m} = 0, \qquad r = \pm i\sqrt{\frac{K}{m}}$$

$$\omega_0^2 = \frac{K}{m}$$ Nat. freq. of vibration.

$$u(t) = A\cos\left(\sqrt{\tfrac{K}{m}}\,t\right) + B\sin\left(\sqrt{\tfrac{K}{m}}\,t\right)$$

$$= A\cos(\omega_0 t) + B\sin(\omega_0 t).$$

To obtain a more useful representation introduce an angle "δ".



$$\tan\delta = \frac{B}{A}$$

$$\sin\delta = \frac{B}{R}$$

$$\cos\delta = \frac{A}{R}$$

$$\Rightarrow \quad B = R\sin\delta$$
$$A = R\cos\delta.$$

$$\Rightarrow \quad u(t) = R\cos\delta\cos(\omega_0 t) + R\sin\delta\sin(\omega_0 t)$$

$$= R\cos(\omega_0 t - \delta)$$

ω_0 → Natural freq. of vibration.

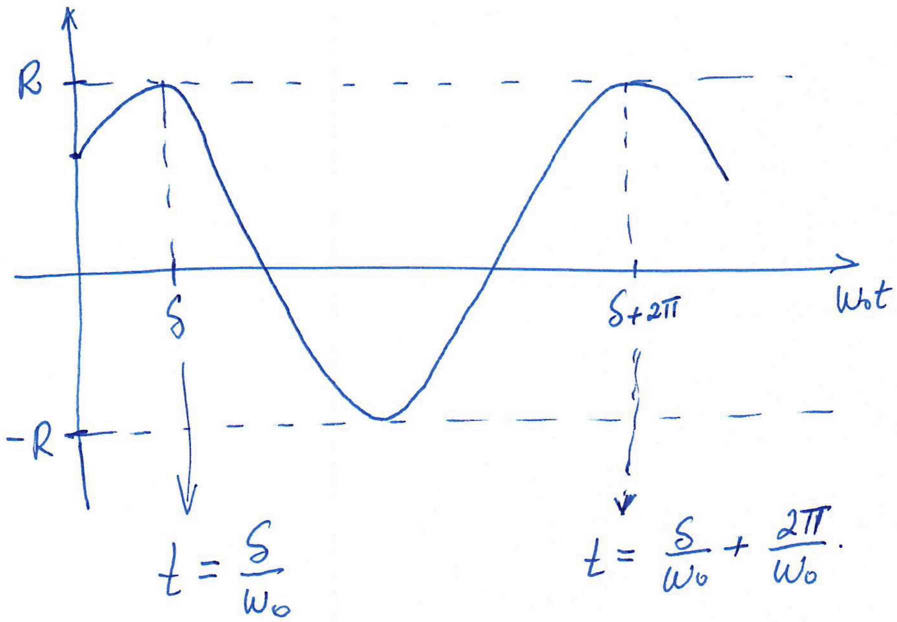
δ → phase angle

R → $\boxed{\text{amplitude of motion}}$

$$\boxed{\text{period} = \frac{2\pi}{\omega_0} = T}$$

Graph of

$$u(t) = R\cos(\omega_0 t - \delta).$$



This behavior is harder to predict from the original form of the Solution

$$u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t).$$

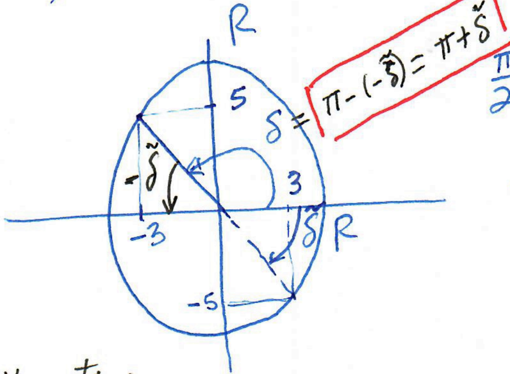
(II) If $u(t) = \underbrace{-3}_{=A}\cos(2t) + \underbrace{5}_{=B}\sin(2t)$

$$\tan(\delta) = \frac{5}{-3}$$

$$\Rightarrow \arctan\left(\frac{5}{-3}\right) = \underbrace{\arctan\left(\frac{-5}{3}\right)}_{=\tilde{\delta}} \simeq -1.03$$

Using Scient. Calculator.

Now,



$\frac{\pi}{2} < \delta < \pi$ 2nd quadrant

$-\frac{\pi}{2} < \tilde{\delta} < 0$ 4th quadrant.

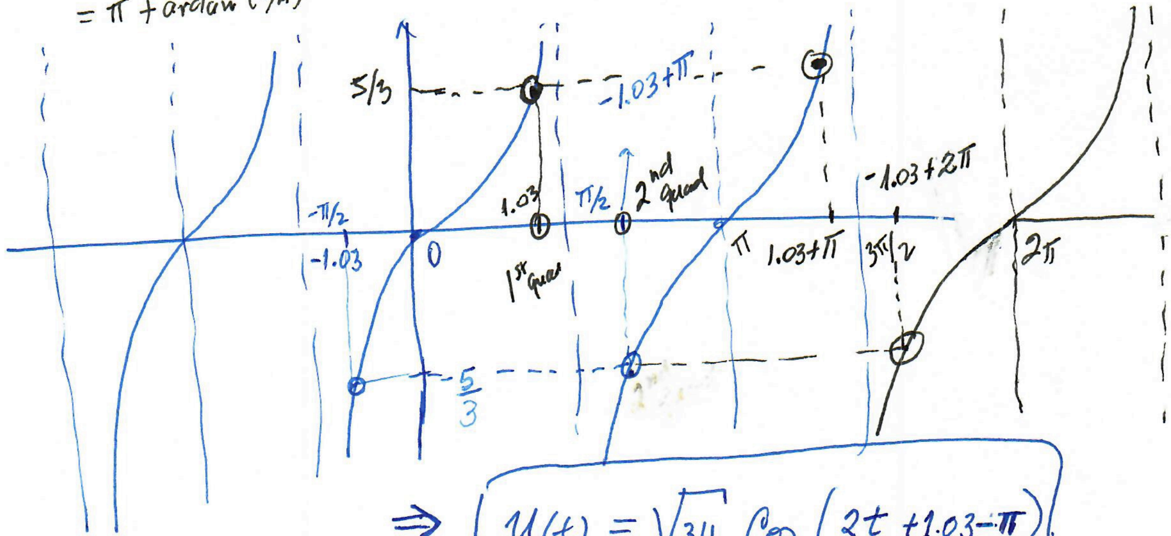
$\tilde{\delta} = -1.03$

$\tilde{\delta} < 0 \Rightarrow \boxed{\delta = \tilde{\delta} + \pi.}$

Alternative:

Directly: $\pi - \arctan(5/3) = \pi - 1.03$.
$= \pi + \arctan(5/-3)$
$= \pi + \arctan(B/A)$

$\Rightarrow \delta = -1.03 + \pi$ (in 2nd quad.)



$$\Rightarrow \boxed{u(t) = \sqrt{34}\cos(2t + 1.03 - \pi)}$$

So $A < 0$, $B > 0$

$$\Rightarrow \delta = \pi + \arctan\left(\frac{B}{A}\right).$$

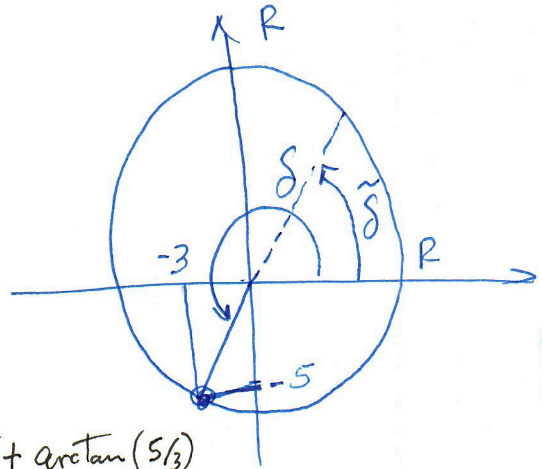
III

If $u(t) = -3\cos 2t - 5\sin 2t$ $\quad A<0,\ B<0.$

$$\tan\delta = \frac{-5}{-3} = \frac{5}{3} \Rightarrow \tilde{\delta} = \arctan\left(\frac{5}{3}\right) \doteq 1.03$$

Using scientific calculator

$0 < \tilde{\delta} < \pi/2$



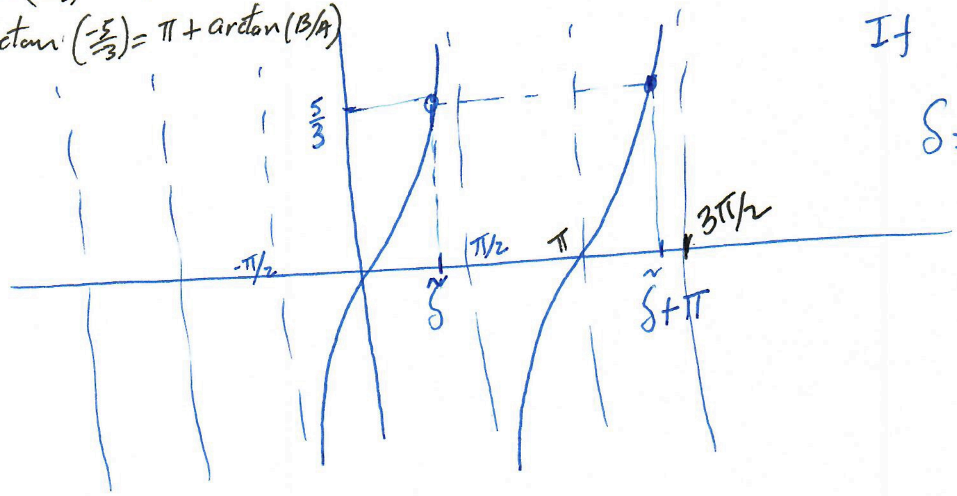
$\pi < \delta < 3\pi/2$ $\quad 3^{rd}$ quadrant.

$0 < \tilde{\delta} < \pi/2$ $\quad 1^{st}$ quadrant

$$\boxed{\delta = \pi + \tilde{\delta}} \Rightarrow u(t) = \sqrt{34}\cos(2t - 1.03 - \pi)$$

$\pi + \arctan(5/3)$

$= \pi + \arctan\left(\frac{-5}{-3}\right) = \pi + \arctan(B/A)$



If $A<0,\ B<0$

$$\delta = \pi + \arctan\left(\frac{B}{A}\right)$$

IV

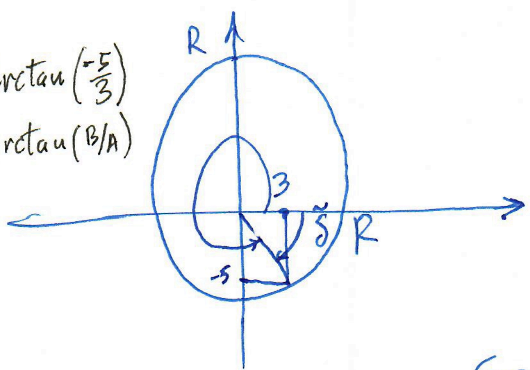
If $u(t) = 3\cos(2t) - 5\sin(2t)$, $\quad A>0,\ B<0$

4^{th} quadrant

$$\tan\delta = \left(\frac{-5}{3}\right) \Rightarrow \tilde{\delta} = \arctan\left(\frac{-5}{3}\right) \simeq -1.03$$

$\arctan\left(\frac{-5}{3}\right)$

$= \arctan(B/A)$



$-\frac{\pi}{2} < \tilde{\delta} < 0$

$\Rightarrow \delta = \tilde{\delta} + 2\pi$

$\Rightarrow u(t) = \sqrt{34}\cos(2t + 1.03)$

However,

$$\cos(\omega_0 t + \delta) = \cos(\omega_0 t - \tilde{\delta} - 2\pi) =$$

$$= \cos(\omega_0 t - \tilde{\delta})$$

So it's ok to use $\tilde{\delta}$ instead of $\delta = \tilde{\delta} + 2\pi$

Summarizing:

If $\quad u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$

$$\Rightarrow u(t) = R\cos(\omega_0 t - \delta)$$

where $\quad R = \sqrt{A^2+B^2}$

and

$$\delta = \begin{cases} \arctan(B/A), & \text{if } A \geq 0,\ B \geq 0. \text{ I quad.} \\ \arctan(B/A) + \pi, & \text{if } A \leq 0,\ B \geq 0. \text{ II quad.} \\ \arctan(B/A) + \pi, & \text{if } A \leq 0,\ B \leq 0. \text{ III quad.} \\ \arctan(B/A) + 2\pi, & \text{if } A \geq 0,\ B \leq 0, \text{ IV quad.} \end{cases}$$

However,

$$u(t) = R\cos(\omega_0 t - \delta) = R\cos\left(\omega_0 t - \left(\arctan\left(\frac{B}{A}\right) + 2\pi\right)\right)$$

$$= R\cos\left(\omega_0 t - \arctan\frac{B}{A} - 2\pi\right) =$$

$$= R\cos\left(\omega_0 t - \arctan\frac{B}{A}\right)$$

Underdamped Motion. Quasi-frequency and quasi-period.

General Solution: (Small damping $\Leftrightarrow \gamma^2-4km<0$)

$$u(t) = A\, e^{-\gamma t/2m} \cos(\mu t) + B\, e^{-\frac{\gamma}{2m}t} \sin(\mu t).$$

$$= R\, e^{-\frac{\gamma}{2m}t} \cos(\mu t - \delta)$$

If

$$m u'' + \gamma u' + k u = 0.$$

$$u'' + \frac{\gamma}{m} u' + \frac{k}{m} u = 0$$

$$r^2 + \frac{\gamma}{m} r + \frac{k}{m} = 0 \Rightarrow r = -\frac{\gamma}{2m} \pm \frac{\sqrt{\frac{\gamma^2}{m^2} - 4\frac{k}{m}}}{2}$$

$$\text{or} \quad r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$$

$\boxed{\gamma^2 - 4km < 0}$

then $\boxed{\mu \equiv \frac{\sqrt{4km - \gamma^2}}{2m}} = \left[\frac{4km - \gamma^2}{4m^2} \right]^{1/2} = \left[\frac{k}{m} - \frac{\gamma^2}{4m^2} \right]^{1/2}$ quasi-frequency.

Relationship with natural freq.

$$\frac{\mu}{\omega_0} = \frac{[\quad]^{1/2}}{\left[\frac{k}{m}\right]^{1/2}} = \left[\frac{4km - \gamma^2}{4m^{\cancel{2}} \frac{k}{\cancel{m}}} \right]^{1/2} = \left[\frac{4km - \gamma^2}{4km} \right]^{1/2}$$

$(1-x)^{1/2} \simeq 1 - \frac{x}{2} + \cdots$

$$= \left[1 - \frac{\gamma^2}{4km} \right]^{1/2} \simeq 1 - \frac{\gamma^2}{8km} \Rightarrow \mu < \omega_0$$

Also, $T_d = \frac{2\pi}{\mu}$ quasi-period.

Therefore,

$$\boxed{\text{Quasi-frequency} = \mu = \frac{\sqrt{4km-\gamma^2}}{2m}}$$

It only makes sense when $\gamma^2-4km<0$

underdamped.

$$\boxed{\text{Natural frequency} = \omega_0 = \left(\frac{k}{m}\right)^{1/2}}$$

This is the case when $\gamma=0$ (No damping).

Relationship between these two frequencies:

$$\frac{\mu}{\omega_0} = \left[1-\frac{\gamma^2}{4km}\right]^{1/2} \simeq 1-\frac{\gamma^2}{8km}$$

or

$$\boxed{\mu \simeq \omega_0 - \frac{\gamma^2}{8km}\omega_0 = \omega_0\left(1-\frac{\gamma^2}{8km}\right)}$$

$$\left[\frac{1}{1-x}\right] = 1+x+x^2+\cdots+x^4+\cdots$$

$$\boxed{\frac{\omega_0}{\mu} = \frac{1}{1-\gamma^2/8km} = 1+\frac{\gamma^2}{8km}}$$

We can also define

$$\boxed{\text{Quasi-Period} = T_d = \frac{2\pi}{\mu}}$$

Relationship between period and quasi-period.

$$\frac{T_d}{T} = \frac{2\pi/\mu}{2\pi/\omega_0} = \frac{\omega_0}{\mu} = \left(1-\frac{\gamma^2}{4km}\right)^{-1/2} \simeq 1+\frac{\gamma^2}{8km}$$

So $T_d = T\left(1+\frac{\gamma^2}{8km}\right)$