3.3 Complex Roots of the characteristic Equation

We continue our derivation of Solutions for the 2nd order Constant coefficients linear equation and homogeneous given by

$$\boxed{a y''(t) + b y'(t) + c y(t) = 0} \quad (3.3.1)$$

We have already shown that (3.3.1) will have a soln such as

$$y(t) = e^{rt}, \quad r = ?$$

if r satisfies the algebraic equation

$$\boxed{ar^2 + br + c = 0.} \qquad (3.3.2)$$

Therefore, "r" can be obtained from

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $\boxed{b^2 - 4ac < 0}$, then we have complex roots.

$$\boxed{r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta} \quad (3.3.3)$$

Example:

$$\boxed{y'' + 3y' + 4y = 0} \qquad (3.3.4)$$

Charact. Equ.:

$$r^2 + 3r + 4 = 0$$

$$r_{1,2} = \frac{-3 \pm \sqrt{9-16}}{2} = \frac{-3 \pm \sqrt{7}\,i}{2}$$

Complex roots

Therefore, the "formal" solutions would be

$$\boxed{y_1(t) = e^{\left(-\frac{3}{2} + i\frac{\sqrt{7}}{2}\right)t}}, \quad \boxed{y_2(t) = e^{\left(-\frac{3}{2} - i\frac{\sqrt{7}}{2}\right)t}} \qquad (3.3.5)$$

At this point, we need a definition of

$$\boxed{e^{(\alpha + i\beta)t} = ?}$$

We start by defining "Euler's formula":

$$\boxed{e^{it} \overset{def}{=} \cos t + i \sin t} \qquad (3.3.6)$$

Then,

$$e^{i(-t)} = \cos(-t) + i\sin(-t) = \cos t - i\sin t$$

$$e^{i\beta t} = \cos(\beta t) + i\sin(\beta t)$$

and we also define

$$e^{(\alpha+i\beta)t} = e^{\alpha t+i\beta t}$$

$$\stackrel{\text{def.}}{=} e^{\alpha t}\cdot e^{i\beta t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$$

or

$$\boxed{e^{(\alpha+i\beta)t} \stackrel{\text{def.}}{=} e^{\alpha t}\cos\beta t + i\, e^{\alpha t}\sin\beta t.} \qquad (3.3.7).$$

Therefore our "formal solutions" (3.3.5).
Can be written as

$$\boxed{y_1^c(t) = e^{\left(-\frac{3}{2}+i\frac{\sqrt{7}}{2}\right)t} = e^{-\frac{3}{2}t}\cos\left(\frac{\sqrt{7}}{2}t\right) + i\, e^{-\frac{3}{2}t}\sin\left(\frac{\sqrt{7}}{2}t\right)}$$

$$y_2^c(t) = e^{-\frac{3}{2}t}\cos\left(-\frac{\sqrt{7}}{2}t\right) + i\, e^{-\frac{3}{2}t}\sin\left(-\frac{\sqrt{7}}{2}t\right) \quad (3.3.6)$$

or

$$\boxed{y_2^c(t) = e^{-\frac{3}{2}t}\cos\left(\frac{\sqrt{7}}{2}t\right) - i\, e^{-\frac{3}{2}t}\sin\left(\frac{\sqrt{7}}{2}t\right)}$$

From (3.3.7), it is possible to show that (3.3.7)

$$\boxed{\left(e^{(\alpha+i\beta)t}\right)' = (\alpha+i\beta)\, e^{(\alpha+i\beta)t}}$$

and verify that $y_1(t)$ and $y_2(t)$ are indeed solns. of (3.3.4).

Now, we use thm 3.2.6 in book to show that the real part and imaginary part of (3.3.6) and (3.3.7) are also solns of (3.3.4).

<u>thm 3.2.6</u>

Consider : $y'' + p(t)y' + q(t)y = 0$ (4.1)

p, q conts in (α, β). If

$$y(t) = u(t) + iV(t)$$

is a complex soln.. Then its real part $u(t)$ and imaginary part $v(t)$ are also solns.

Proof. Since $y'(t) = u'(t) + i\,v'(t)$

$$y''(t) = u''(t) + i\,v''(t)$$

Subst. into (4.1) separating the real and imaginary part leads to

$$(u'' + p(t)u' + q(t)u) + i\,(v'' + p(t)v' + q(t)v) = 0$$

$$\Rightarrow \quad u'' + p(t)u' + q(t)u = 0$$

$$v'' + p(t)v' + q(t)v = 0.$$

As a consequence,

$$y_1(t) = e^{-\frac{3}{2}t}\cos\left(\frac{\sqrt{7}}{2}t\right) \quad \text{and} \qquad (5.1)$$

$$y_2(t) = e^{-\frac{3}{2}t}\sin\left(\frac{\sqrt{7}}{2}t\right) \qquad (5.2)$$

are real solns. of (3.3.4).

By evaluating the Wronskian

$$W[y_1, y_2](t) \neq 0.$$

Therefore, (5.1)-(5.2) form a fund. set of solns for (3.3.4).