3.2 Solutions of Linear Homogeneous Equs.; the Wronskian.

Is ~~(8.3)~~ (9.0) the only solution of IVP (5.1)–(5.2)?

Thm 3.2.1 Consider the IVP:

$$\begin{cases} y''(t) + p(t)y'(t) + q(t)y(t) = g(t) & (1) \\ y(t_0) = y_0, \quad y'(t_0) = y_0' & (2) \end{cases}$$

If a) p, q, g are conts. on (α, β)

b) $t_0 \in (\alpha, \beta)$

$\Rightarrow$ IVP. (1)–(2) has ① a unique ② solution on ③ (α, β).

Proof.– (Not easy).

Existence.

Uniqueness: Consider first that the theorem is true for

in general homogeneous IVP.

$$\begin{cases} y''(t) + p(t)y'(t) + q(t)y(t) = 0 & (3) \\ y(t_0) = 0, \quad y'(t_0) = 0 \end{cases}$$

then an obvious solution is $y(t) \equiv 0$.

Uniqueness implies this is the only one.

Now, Consider the general nonhomogeneous IVP:

$$\begin{cases} y''(t) + p(t)\,y'(t) + q(t)\,y(t) = g(t) & (1) \\ y(t_0) = y_0, \quad y'(t_0) = y_0' & (2) \end{cases}$$

If there were two solutions $y_1(t)$ and $y_2(t)$ of (1) and (2), then

$y_1(t) - y_2(t)$ Satisfies the homogeneous problem

(3) and (4). In fact,

$$(y_1(t) - y_2(t))'' + p(t)\,(y_1 - y_2)' + q(t)\,(y_1 - y_2) =$$

$$= y_1'' + p y_1' + q y_1 + y_2'' + p y_2' + q y_2 = g(t) - g(t) = 0 \checkmark$$

Also $y_1(t_0) - y_2(t_0) = y_0 - y_0 = 0. \checkmark$

$y_1'(t_0) - y_2'(t_0) = y_0' - y_0' = 0 \checkmark$

Therefore, $y_1(t) - y_2(t) \equiv 0 \Rightarrow \underline{y_1(t) = y_2(t)}$ for all t.

Thm 3.2.2. (Superposition Pple)

If $y_1(t)$ and $y_2(t)$ are two solutions of

$$y'' + py' + qy = 0$$

then $c_1 y_1 + c_2 y_2$ is also a soln. for any c_1, c_2.

Proof.- (Easy).

Summarizing Theorems in Sections 3.2

If $y_1(t)$ and $y_2(t)$ are two Solutions of

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0 \qquad (11.1)$$

$p(t), q(t)$ Conts in (α,β).

and Wronskian of y_1 and y_2, is given by

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \qquad (11.2)$$

then,

Thm 3.2.3 It is possible to find C_1 and C_2 Such that

$$C_1 y_1(t) + C_2 y_2(t)$$

Satisfies (11.1) and arbitrary ICs: $\begin{cases} y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$

if and only if

$$W(y_1, y_2)(t_0) \neq 0$$

Thm 3.2.4 The family of Solutions

$$y(t) = C_1 y_1(t) + C_2 y_2(t), \quad C_1, C_2 \text{ arbitrary}$$

is the general solution of (11.1) (Every soln. is included) if and only if there is t^* Such that

$$W(y_1, y_2)(t^*) \neq 0, \quad t^* \text{ in } (\alpha,\beta)$$

<u>Definition</u> The Solutions $y_1(t)$ and $y_2(t)$ are called a <u>fundamental set of Solutions</u> if

$$W(y_1,y_2)(t^*) \neq 0 \quad \text{at some } t^* \text{ in } (\alpha,\beta)$$

<u>Examples:</u> a) $y''+2y'-15y=0$

$y_1(t)=e^{-5t}$, $y_2(t)=e^{3t}$ fund. set Solns.

Since $$W(y_1,y_2)(t)=\begin{vmatrix} e^{-5t} & e^{3t} \\ -5e^{-5t} & 3e^{3t}\end{vmatrix} = 8e^{-2t}\neq 0$$ for all t.

b) $y''-6y'+9y=0$

Two solns: $y_1(t)=e^{3t}$, $y_2(t)=4e^{3t}$

But $$W(y_1,y_2)(t)=\begin{vmatrix} e^{3t} & 4e^{3t} \\ 3e^{3t} & 12e^{3t}\end{vmatrix} = 0,$$ for all t.

Then, $y_1(t)$ and $y_2(t)$ does not form a fund. set of solns.

c) $t^2y''(t)+ty'(t)+y=0$, $t>0$

Solns: $y_1(t)=\cos(\ln t)$, $y_2(t)=\sin(\ln t)$ (Fund set of solns.)

and $$W(y_1,y_2)(t)=\begin{vmatrix} \cos(\ln t) & \sin(\ln t) \\ -\dfrac{\sin(\ln t)}{t} & \dfrac{\cos(\ln t)}{t}\end{vmatrix}$$

$$=\frac{\cos^2(\ln t)}{t}+\frac{\sin^2(\ln t)}{t}=\frac{1}{t}\neq 0,\quad t>0$$

Two Questions:

1) **Theoretical:** Given

$$y''(t) + p(t)\,y'(t) + q(t)\,y(t) = 0, \quad p(t), q(t) \text{ conts in } (\alpha,\beta)$$

Is it always possible to find $y_1(t)$ and $y_2(t)$ forming a fund. set of solns?

2) **Practical:** In case (1) is true

How can we find $y_1(t)$ and $y_2(t)$?

Answer to question (1) is provided by

Thm 3.2.5

For $y''(t) + p(t)y'(t) + q(t)y = 0$

with $p(t)$ and $q(t)$ conts. on (α,β),

there is always a fund. set of solns. $\{y_1(t), y_2(t)\}$

where $y_1(t)$ satisfies and $y_2(t)$ satisfies

$$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = 1 \\ y'(t_0) = 0 \end{cases} \qquad \begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = 0 \\ y'(t_0) = 1 \end{cases}$$

Discuss Proof

The answer to Question (2) has been partially given for constant coefficient equations

$$ay'' + by' + cy = 0$$

Look for solns. in the form: $y(t) = e^{rt}$

$$ar^2 + br + c = 0 \qquad (14.1)$$

i) If (14.1) has two different real roots r_1 and r_2, $r_1 \neq r_2$

then $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$

form a fund. set of solns, since

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1) e^{(r_1 + r_2)t} \neq 0, \text{ if } r_2 \neq r_1.$$

ii) If (14.1) has no real roots, but two conjugate complex roots

$r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$?

iii) If (14.1) has only one root (real) ?

In the next sections, we provide answer to these two cases.

Thm 3.2.7 (Abel's thm)

i) y_1 and y_2 Solns of

$$y'' + p(t)y' + q(t)y = 0$$

ii) p, q conts on (α, β).

Then

$$W(y_1, y_2)(t) = c\, e^{-\int p(t)\,dt}, \quad c. \text{ constant}$$

Notice that based on thm 3.2.7

$W(y_1, y_2)(t)$ is always zero (for all t)

or is never zero.

Proof Since y_1 and y_2 are Solns. they Satisfy

$$\left(y_1'' + p(t)y_1' + q(t)y_1 = 0\right)\ (-y_2)$$

$$\left(y_2'' + p(t)y_2' + q(t)y_2 = 0\right)\ (y_1)$$

$$\left(-y_1''y_2 + y_2''y_1\right) + p(t)\underbrace{\left(-y_2y_1' + y_1y_2'\right)}_{\|} = 0 \quad (A.1)$$

Notice

$$W(y_1,y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

and $[W(y_1,y_2)(t)]' = \cancel{y_1'(t)y_2'(t)} + y_1(t)y_2''(t) - \cancel{y_2'(t)y_1'(t)} - y_2(t)y_1''(t)$

Therefore, (A.1) Can be written in terms of the Wronskian as

$$W'(t) + p(t)W(t) = 0$$

Linear 1st order

$$\mu(t) = e^{\int p(t)dt}$$

then

$$\frac{d}{dt}\left[W(t)\, e^{\int p(t)dt}\right] = 0$$

or $\int dt$:

$$\boxed{W(t) = C\, e^{-\int p(t)dt}}$$

Sect 3.2

#30 Find Wronskian

$$(\cos t)y'' + (\sin t)y' - ty = 0$$

$$W(y_1, y_2)(t) = c\, e^{-\int p(t)\,dt} \quad \text{Abel's thm.}$$

then

$$y'' + \frac{\sin t}{\cos t} y' - \frac{t}{\cos t} y = 0$$

$u = \cos t$
$du = -\sin t\, dt$

and

$$W(y_1, y_2)(t) = c\, e^{-\int \frac{\sin t}{\cos t} dt} = c\, e^{\ln(|\cos t|)} = c \cos t$$