CHAPTER 3

Second order Linear Equations.

$$y''(t) = f\,(t, y(t), y'(t)) \qquad (1.1)$$

3.1 Homogeneous Equations with constant Coefficients.

Consider

	Non-trivial Solns.
a) $y''(t) - y(t) = 0$ $y'' = y$	$y_1(t) = e^t$, $y_2(t) = e^{-t}$
b) $y'' + y = 0$ $y'' = -y$	$y_1(t) = \sin t$, $y_2(t) = \cos t$
c) $y'' - y' = 0$ $y'' = y'$	$y_1(t) = e^t$, $y_2(t) \equiv 1$
d) $y'' + y' = 0$ $y'' = -y'$	$y_1(t) = e^{-t}$, $y_2(t) \equiv 1$.

Notice that all these equations have exponentials solns.

Even $y_1(t) = \sin t$ and $y_2(t) = \cos t$

Can be written as

$$\boxed{y_1(t) = \sin t = \frac{e^{it} - e^{-it}}{2i}}$$

Since $\dfrac{e^{it} - e^{-it}}{2i} = \dfrac{\cos t + i\sin t - (\cos t - i\sin t)}{2i} = \dfrac{2\cancel{i}\sin t}{2\cancel{i}} = \sin t$

Also, $\boxed{y_2(t) = \cos t = \dfrac{e^{it} + e^{-it}}{2}}$

All these eqns. are particular cases of

$$\boxed{a y''(t) + b y'(t) + c y(t) = 0} \qquad (2.1)$$

which is a particular case of

$$\boxed{P(t)\, y''(t) + Q(t)\, y'(t) + R(t)\, y(t) = G(t)} \qquad (2.2)$$

or if $P(t) \neq 0$

$$\boxed{y''(t) + p(t)\, y'(t) + q(t)\, y(t) = g(t).} \qquad (2.3)$$

where $p(t) = \dfrac{Q(t)}{P(t)}, \quad q(t) = \dfrac{R(t)}{P(t)}$

$$g(t) = \frac{G(t)}{P(t)}$$

i) Equ. (2.2) is the more general form of <u>Second order linear equations</u>

a) If $G(t) \equiv 0$, this equation is called homogeneous.

b) If $G(t) \not\equiv 0$, this equ. is called nonhomogeneous.

ii) Equ. (2.1) is called: Second order linear homogeneous <u>ordinary differential equation</u> (ODE) with constant coefficients.

iii) If rhs of (2.1) is $g(t) \neq 0$

then

$$a y''(t) + b y'(t) + c y(t) = g(t)$$

is called: Sec. order linear nonhomogeneous ODE with constant coefficients

The IVP for 2^{nd} order ODEs requires two initial conditions (ICs). For example,

$$\underline{\text{IVP}}: \begin{cases} P(t)\, y''(t) + Q(t)\, y'(t) + R(t)\, y(t) = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$$

General Solutions of

$$a\,y''(t) + b\,y'(t) + c\,y(t) = 0. \qquad (4.1)$$

a, b, c constants.

Based on our previous experience with equs. (a)–(d). we will look for Solutions in the form:

$$\boxed{y(t) = e^{rt}},$$

r to be determined.

$y'(t) = re^{rt}$, $y''(t) = r^2e^{rt}$

Substituting into (4.1).

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$$

or

$$e^{rt}(ar^2 + br + c) = 0$$

So to determine "r" we need to find the roots of

$$ar^2 + br + c = 0$$

Three cases arise:

a) $r_1 \neq r_2$, r_1, r_2 in $\mathbb{R}$. Sect 3.1

b) $r_1 = r_2$, r_1, r_2 in $\mathbb{R}$. Sect 3.4

c) No real roots Sect 3.3
r_1, r_2 conjugate complex numbers.

Example: Solve the IVP:

$$\begin{cases} y''(t) + 2y'(t) - 15y(t) = 0 & (5.1) \\ y(0) = 6, \quad y'(0) = -14 & (5.2) \end{cases}$$

Answ:

Look for Solns. in the form of

$$y(t) = e^{rt}, \quad \text{"}r\text{" to be determined}$$

After Substitution

$$e^{rt}\left(r^2 + 2r - 15\right) = 0 \Rightarrow \underbrace{\boxed{r^2 + 2r - 15 = 0}}_{\text{Characteristic polynomial.}} \quad (5.3)$$

Therefore, $(r+5)(r-3) = 0$

or $r_1 = -5, \; r_2 = 3$ are the roots

As a result, two Solutions are

$$y_1(t) = e^{-5t}, \quad y_2(t) = e^{3t}.$$

It can be easily verified that

$$\boxed{y(t) = c_1 e^{-5t} + c_2 e^{3t}} \quad (5.4)$$

is also a Solution

Therefore, the linear combination of $y_1(t) = e^{-5t}$ and $y_2(t) = e^{3t}$ is also a Soln. Thus, (5.4) is a family of infinite Solutions.

<u>Idea</u>: Will it be possible to find the

a) Soln. of our IVP from this family of Solns?

or

b) Are there constants "C_1" and "C_2" Such that

$$y(t) = C_1 e^{-5t} + C_2 e^{3t}$$

also satisfies the IC's

$y(0) = 6$ and $y'(0) = -14$. ?

Well, let us try!

Compute first $y'(t)$

$$y'(t) = -5c_1 e^{-5t} + 3c_2 e^{3t}$$

Then

$$\begin{aligned} y(0) = 6 &\iff \\ y'(0) = -14 &\iff \end{aligned} \begin{cases} c_1 + c_2 = 6 \\ -5c_1 + 3c_2 = -14 \end{cases}$$

Algebraic Linear System of two Eqns and two unknowns.

How do we solve it?

Many ways!

One way: Consider the associated matrix

$$A = \begin{pmatrix} 1 & 1 & 6 \\ -5 & 3 & -14 \end{pmatrix}$$

row reduce it

$$A \sim \begin{pmatrix} 1 & 1 & 6 \\ 0 & 8 & 16 \end{pmatrix}$$

Thus

$$8c_2 = 16 \Rightarrow c_2 = 2$$

$$c_1 + c_2 = 6 \Rightarrow c_1 = 6 - c_2 = 6 - 2 = 4$$

Therefore, a Soln. of the IVP (5.1)-(5.2) is given by

$$y(t) = 4e^{-5t} + 2e^{3t}$$

Now, consider the more general IVP:

$$\begin{cases} y''(t) + p(t)\,y'(t) + q(t)\,y(t) = 0 & (9.1) \\ y(t_0) = y_0, \quad y'(t_0) = y_0' & (9.2) \end{cases}$$

Questions of the Soul?

Inspired Questions?

1) If $y_1(t)$ and $y_2(t)$ are two different Solutions of (9.1), is it possible (always) to find "C_1" and "C_2" such that

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

also Satisfies (8.2) ICs. ?

2) If $y_1(t)$ and $y_2(t)$ are two different Solutions of (9.1).

Is it true that

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

includes every Soln. of (9.1). ?

3) For equation (9.1), is it always possible to find two different solns.

$y_1(t)$ and $y_2(t)$ Such that (1) and (2) are true ?