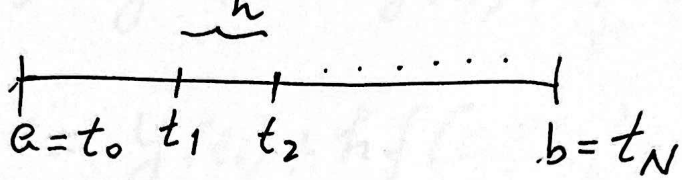
# Euler's Method

Initial Value Problem (IVP)
$$\begin{cases} \dfrac{dy}{dt} = f(t,y), & a \le t \le b \quad (2.1) \\ y(a) = \alpha & \quad (2.2) \end{cases}$$

Mesh :




Equally distributed:

$$h = t_{i+1} - t_i = \frac{b-a}{N}$$

$N+1$ points : $t_0, t_1, \ldots, t_N$

$N$ Subintervals : $(t_0, t_1), (t_1, t_2), \ldots, (t_{N-1}, t_N)$

## Derivation of Euler Method.

Assume $y(t)$ is a unique soln. for (2.1)-(2.2).
And it has two conts. derivatives on $[a,b]$
Expanding $y(t)$ around $t = t_i$ using Taylor's thm.

$$y(t) = y(t_i) + (t - t_i) y'(t_i) + \frac{(t-t_i)^2}{2} y''(\xi_i)$$

for $\xi$ in $(t, t_i)$

In particular, for $i=0,1,2,\ldots,N-1$.

$$y(t_{i+1}) = y(t_i) + \overbrace{(t_{i+1}-t_i)}^{=h} y'(t_i) + \frac{\overbrace{(t_{i+1}-t_i)}^{=h}{}^2}{2} y''(\xi_i)$$

or

$$y(t_{i+1}) = y(t_i) + h\, y'(t_i) + \frac{h^2}{2} y''(\xi_i)$$
$$= y(t_i) + h f(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i) \quad (3.1)$$

by calling $w_i \simeq y(t_i)$, $i=0,1,2,\ldots,N$.

we obtain the equations defining Euler's method neglecting the last Term. in (3.1)

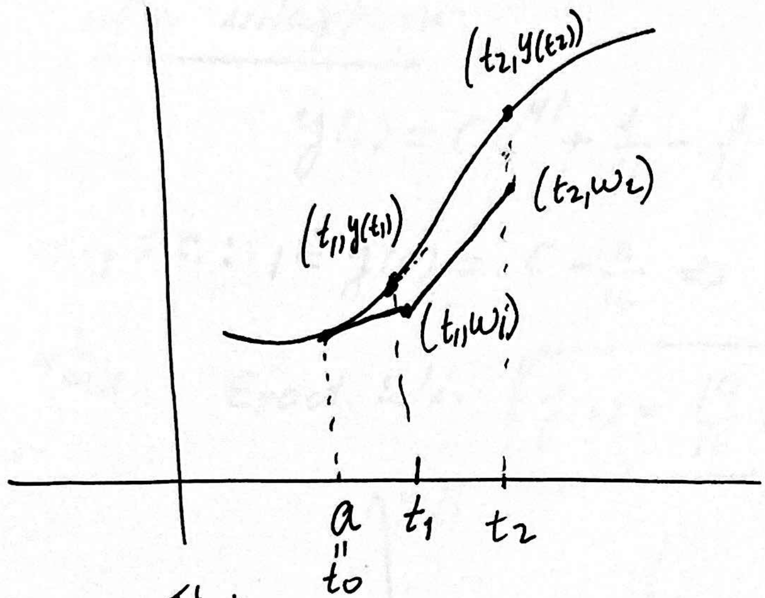
Difference equation
$$\boxed{\begin{cases} w_{i+1} = w_i + h f(t_i, w_i) & (3.2) \\ w_0 = \alpha & (3.3) \end{cases}}$$

for $i=0,1,\ldots,N-1$.

Show Some MATLAB examples.

# Geometrical Interpretation of Euler's Method

$$f(t_i, w_i) \simeq y'(t_i) = f(t_i, y(t_i))$$




$$w_1 = \overset{\substack{\text{If exact} \\ = y_0}}{w_0} + (t_1 - t_0)\overset{\simeq y'(t_0)}{f(t_0, w_0)}$$

$$w_2 = \overset{\simeq y(t_1)}{w_1} + (t_2 - t_1)\overset{\simeq y'(t_1)}{f(t_1, w_1)}$$

IVP

$$\boxed{y' = 1 - t + 4y, \quad y(0) = 1}$$

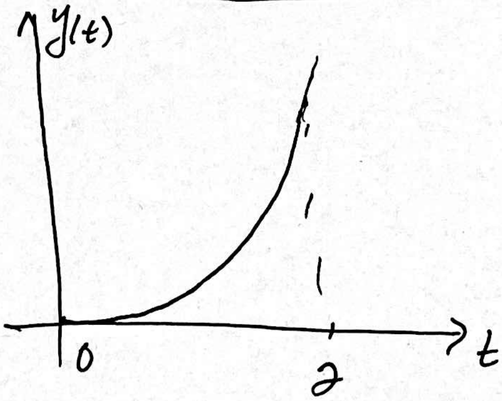
Exact Soln:

After integration:

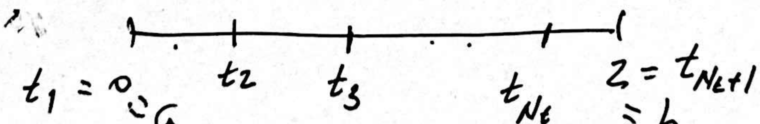
$$y(t) = Ce^{4t} + \frac{t}{4} - \frac{3}{16}$$

Using I.C.: $1 = y(0) = C - \frac{3}{16} \Rightarrow C = \frac{19}{16}$

Then Exact Soln: $\boxed{y(t) = \frac{19}{16} e^{4t} + \frac{t}{4} - \frac{3}{16}}$




Applying Euler's Method in [0, 2].




$h = \Delta t = 0.01 = \frac{b-a}{N_t}$

$N_t = \frac{b-a}{\Delta t}$

$$\boxed{\begin{aligned} w_1 &= 1 \\ w_{i+1} &= w_i + \Delta t\,(1 - t_i + 4w_i) \end{aligned}} \quad i = 1, 2, \ldots, N_t$$

Generating $w_2, w_3, \ldots, w_{N_t+1}$

See the approx. in the MATLAB code in CANVAS.