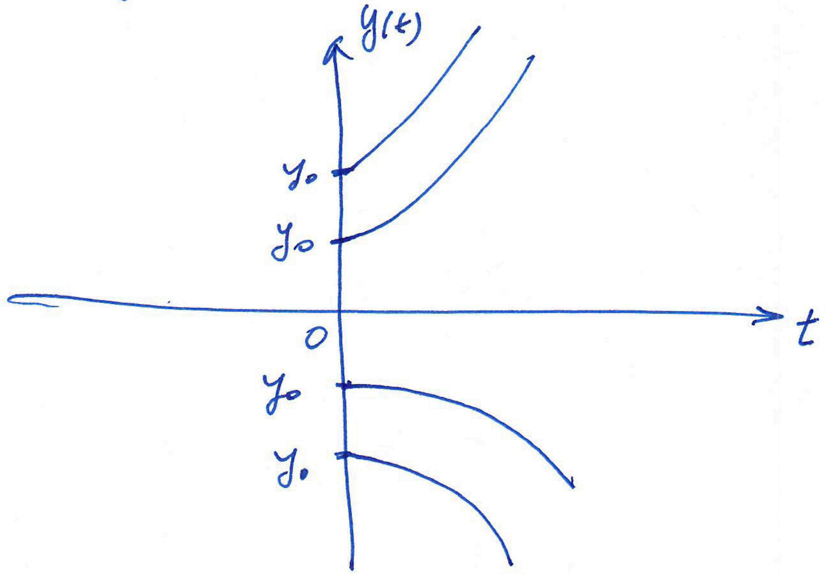
2.5 Autonomous Equations.

1) Exponential growth.

$$\begin{cases} \dfrac{dy}{dt} = y, & y(t) = y_0 e^{t} \\ y(0) = y_0, \end{cases}$$

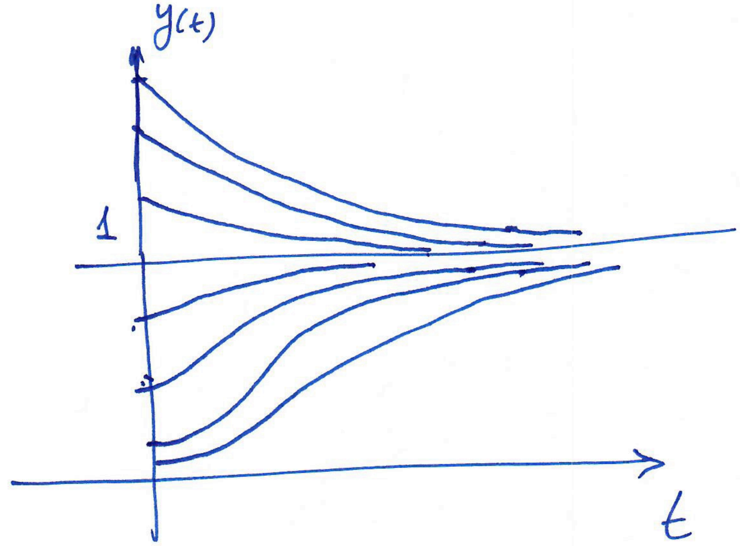
A factor "r" in front of y means a rate of Something: Interest rate, growth rate, flow rate, etc.

2) $$\begin{cases} \dfrac{dy}{dt} = ry, & y(t) = y_0 e^{rt}, \quad r \geq 0 \\ y(0) = y_0, \end{cases}$$



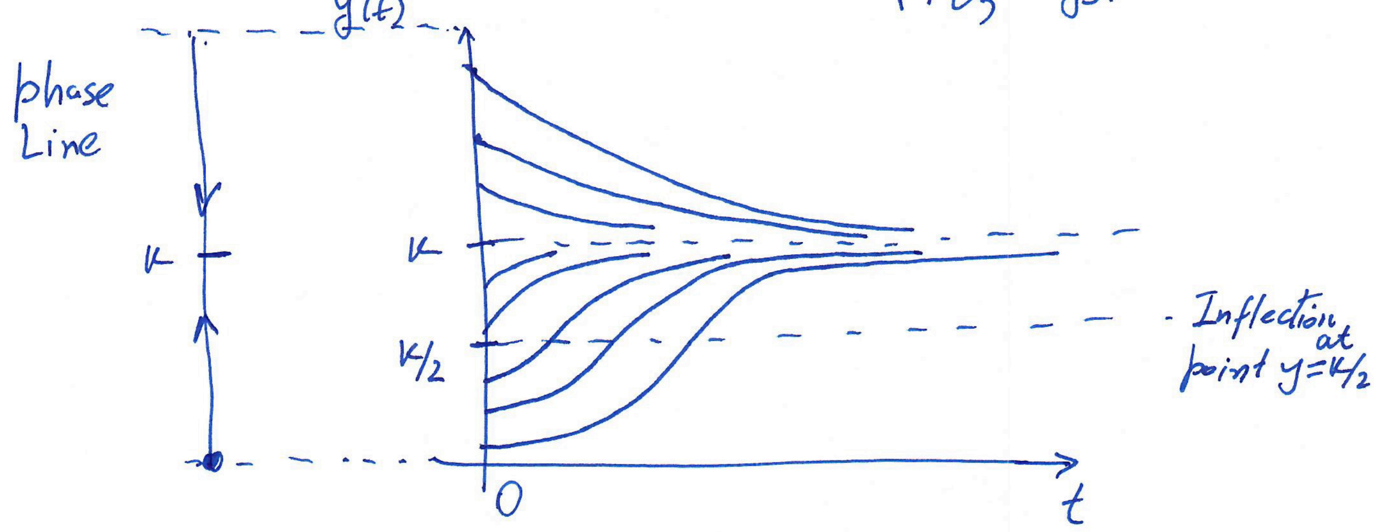
Population model

3) $$\begin{cases} \dfrac{dy}{dt} = r(1-y)y, & r>0, \\ y(0) = y_0 & y_0>0. \end{cases}$$



4) ## Logistic Growth

$$\frac{dy}{dt} = r\left(1-\frac{y}{k}\right)y, \quad k = \text{Saturation constant.}$$
$r>0$, $y_0>0$



Another representation: $\dfrac{dy}{dt}(t) = f(y(t)) \overset{\text{or}}{\underset{\text{Simply}}{=}} f(y)$

where $f(y) = r\left(1-\frac{y}{k}\right)y$.

If a) $0<y<K \Rightarrow y'(t) = r\overset{>0}{\left(1-\frac{y}{K}\right)}\overset{>0}{y} > 0$

$\Rightarrow y(t) \nearrow$.

b) $y>K \Rightarrow y'(t) = r\overset{<0}{\left(1-\frac{y}{K}\right)}\overset{>0}{y} < 0$

$\Rightarrow y(t) \searrow$

If The growth rate $\frac{dy}{dt}(t)$ changes according to the values of $y(t)$.

a) $y'(t)$ is increasing when its slope $y''(t)>0$ and is decreasing when its slope $y''(t)<0$.

So we need to analize

$$y''(t) = \left[r\left(1-\frac{y(t)}{K}\right)y(t)\right]'$$

$$= \frac{d}{dt}\left[f(y(t))\right] = f'(y(t))\frac{dy}{dt} =$$

$$= f'(y)\,f(y).$$

So $$\frac{d^2y}{dt^2}(t) = f'(y(t))\,f(y(t))$$

a) $y''(t)>0$, when $f'(y)$ and $f(y)$ are both positive or both negative

b) $y''(t)<0$, when $f'(y)$ and $f(y)$ have different sign.

Now, $\boxed{f(y) = r\left(1-\frac{y}{k}\right)y}$

then $f'(y) = r\left(1-\frac{y}{k}\right) - \frac{r}{k}y$

$= r - 2\frac{r}{k}y = r\left(1-\frac{2}{k}y\right)$

or $\boxed{f'(y) = r\left(1-\frac{y}{\frac{k}{2}}\right)}$

therefore,

i) If $y = k/2 \Rightarrow f'(y) = 0 \Rightarrow y''(t) = 0.$
for all solutions $y(t)$ of (1.1).

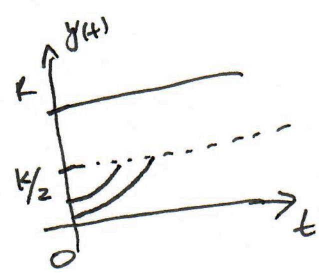
Candidate to inflection points.

ii) If $0 < y < k/2 \Rightarrow f'(y) > 0$ and $f(y) > 0$

$\Rightarrow y''(t) = f(y)f'(y) > 0$

and the growth is accelerated

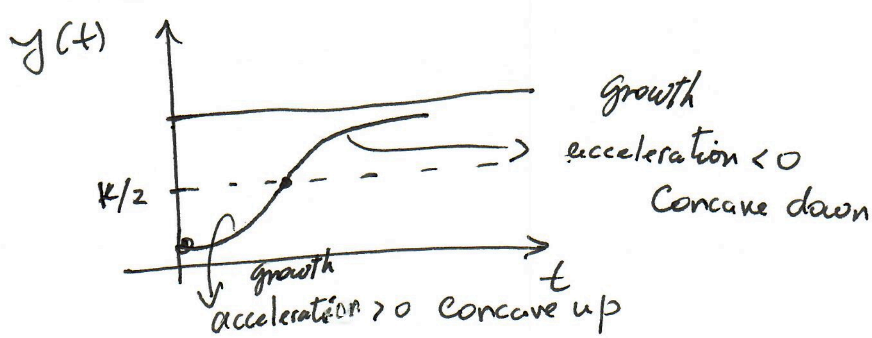
graph is concave up, since $y'(t)\nearrow$.



iii) If $k/2 < y < k \Rightarrow f'(y) < 0$ and $f(y) > 0$

then $y''(t) = f(y)f'(y) < 0$

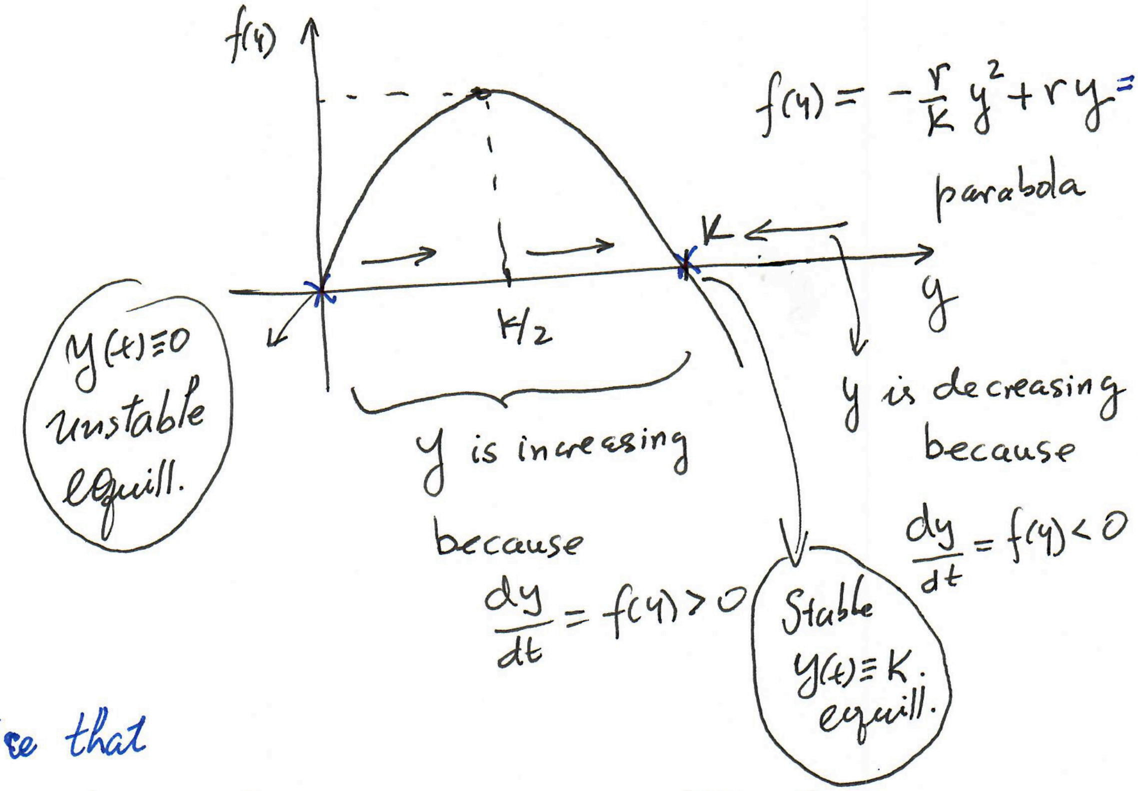
and the growth acceleration is decreasing: $y'(t)\searrow$



The Solns. below (or above) $y(t)=K$ never cross the line $y=K$. This is due to thm 2.4.2 <u>Uniqueness</u> property.

So they converge asymptotically to $y(t)\equiv K$. and they remain in the interval $(0,K)$. or $(K,+\infty)$, respectively.

To help in the visualization of the Solns, it is Convenient to consider the graph of $f(y)$



$$f(y) = -\frac{r}{K}y^2 + ry = ry\left(1-\frac{y}{K}\right)$$

parabola

Notice that

$$f'(y) = -\frac{2r}{K}y + r = 0 \Rightarrow y = \frac{Kr}{2r} = \frac{K}{2}$$

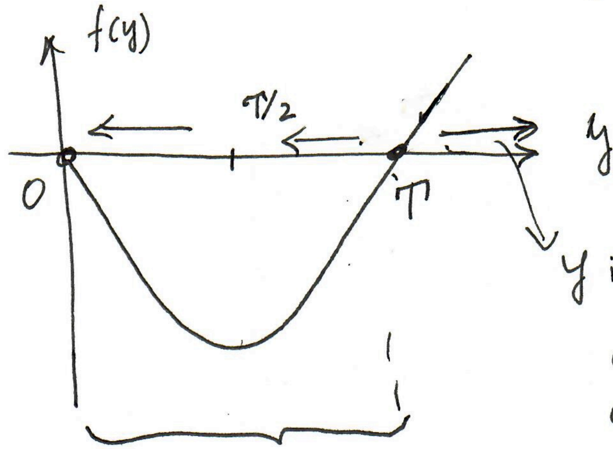
$$= -\frac{2r}{K}\left(y - \frac{K}{2}\right)$$

) Critical Threshold

$$\frac{dy}{dt}(t) = -r\left(1 - \frac{y}{T}\right)y = f(y)$$

$$= \frac{r}{T}y^2 - ry$$

Geometric Analysis



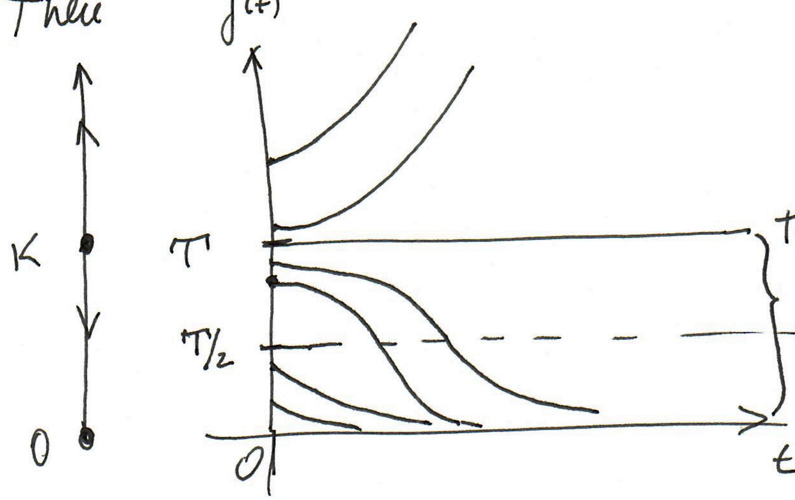
y increases

$\frac{dy}{dt} = f(y) > 0.$

y is decreasing

$\frac{dy}{dt} = f(y) < 0$

then



treshold

population becomes extinct below T ($y < T$).

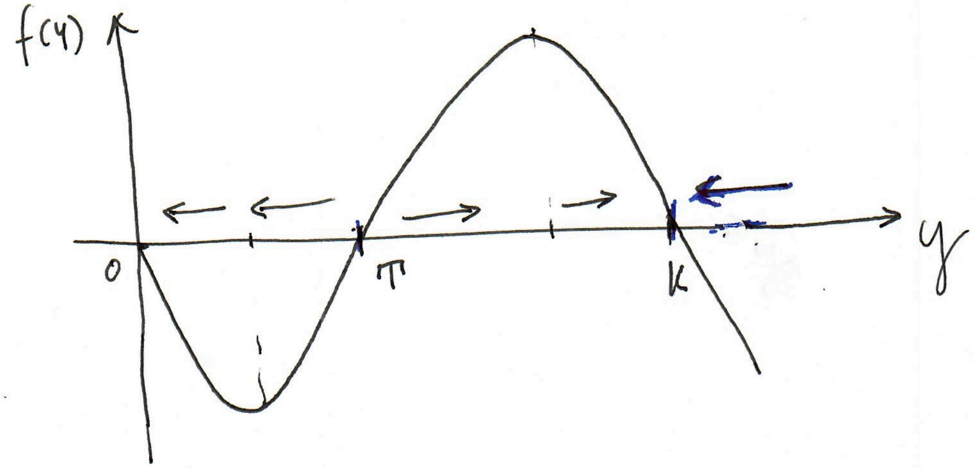
) Logistic Growth with a Treshold.

$$\frac{dy}{dt}(t) = -r\left(1-\frac{y}{T}\right)\left(1-\frac{y}{K}\right)y = f(y)$$

$r>0$

$0<T<K.$

Geometric Analysis



Then,