5.2 / 5.3 Series Solutions near Ordinary points.

Consider

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0 \qquad (1)$$

P, Q, R are polynomials in the neighborhood of a point x_0, with no common factors.

We will look for solns. of (1) as power series about x_0,

$$y(x) = \phi(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \qquad (2)$$

Questions:

① What conditions should equ. (1) satisfy in order to have a soln. as (2)?

② How can we determine the radius of convergence ρ of a soln. of (1) given by a power series as (2)?

Is there an alternative to the ratio test if we do not know the general term of the power series?

Problem 15a. Section 5.2 (after changing variables)

Find a series solution in powers of x $(x_0=0)$.

Of the equation:

$$y'' + x^2 y' + (x^2+2x)y = 0 \qquad (1)$$

Answer: $$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1} n a_n x^{n-1}, \qquad y'' = \sum_{n=2} n(n-1) a_n x^{n-2}$$

Subst. into (1)

$m=n+1 \Rightarrow n=m-1$

$$\sum_{n=2} n(n-1) a_n x^{n-2} + x^2 \sum_{n=1} n a_n x^{n-1}$$

$m=n+2 \Rightarrow n=m-2$

$$+ x^2 \sum_{n=0} a_n x^n + 2x \sum_{n=0} a_n x^n = 0.$$

$m=n-2$
$n=m+2$

$$\sum_{m=0} (m+2)(m+1) a_{m+2} x^m + \sum_{m=2} (m-1) a_{m-1} x^m$$

$$+ \sum_{m=2} a_{m-2} x^m + 2 \sum_{m=1} a_{m-1} x^m = 0 \qquad (2)$$

Equation (2) can be written as

$$2(1)\,a_2 + \left(2a_0 + (3)(2)\,a_3\right)x$$
$$+ \sum_{m=2}\left[(m+2)(m+1)\,a_{m+2} + (m-1+2)\,a_{m-1} + a_{m-2}\right]x^m = 0$$

$1:\quad 2a_2 = 0 \Rightarrow \boxed{a_2 = 0}$

$x:\quad 2a_0 + 6a_3 = 0 \Rightarrow \boxed{a_3 = -\frac{1}{3}a_0}$

$x^m\ (m \geq 2):\quad \boxed{a_{m+2} = \dfrac{-(m+1)\,a_{m-1} - a_{m-2}}{(m+2)(m+1)}, \quad m \geq 2}$

<u>$m=2$</u> $\quad a_4 = \dfrac{-3a_1 - a_0}{(4)(3)} = -\dfrac{a_1}{4} - \dfrac{a_0}{12}$

$m=3 \quad a_5 = \dfrac{-6\cancel{a_2}^{\,0} - a_1}{(5)(4)} = -\dfrac{a_1}{20}.$

$m=4 \quad a_6 = \dfrac{-5a_3 - \cancel{a_2}^{\,0}}{(6)(5)} = -\dfrac{a_3}{6} = -\dfrac{1}{6}\left(-\dfrac{1}{3}a_0\right) = \dfrac{1}{18}a_0$

$$m=5: \quad a_7 = \frac{-6a_4 - a_3}{7.6} = -\frac{a_4}{7} - \frac{a_3}{42} =$$

$$= -\frac{1}{7}\left(\frac{-a_1}{4} - \frac{a_0}{12}\right) - \frac{1}{42}\left(-\frac{1}{3}a_0\right)$$

$$= \frac{a_1}{28} + \left(\frac{1}{84} + \frac{1}{126}\right)a_0$$

or $a_7 = \frac{a_1}{28} + \frac{3+2}{252}a_0$

or $\boxed{a_7 = \frac{a_1}{28} + \frac{5}{252}a_0}$

$$\begin{array}{r|l} 126 & 2 \\ 63 & 3 \\ 21 & 3 \\ 7 & 7 \\ 1 & \end{array} \qquad \begin{array}{r|l} 84 & 2 \\ 42 & 2 \\ 21 & 3 \\ 7 & 7 \\ 1 & \end{array}$$

m.c.m $= 2^2.3^2.7$
$4.9.7$
$= 252$

$252 \lfloor 84$
$= \quad 3$

then,

$$y(x) = a_0 + a_1 x - \frac{1}{3}a_0 x^3 + \left(-\frac{a_1}{4} - \frac{a_0}{12}\right)x^4$$

$$- \frac{a_1}{20}x^5 + \frac{1}{18}a_0 x^6 + \left(\frac{a_1}{28} + \frac{5}{252}a_0\right)x^7 + \cdots$$

$$= a_0\left[1 - \frac{x^3}{3} - \frac{x^4}{12} + \frac{x^6}{18} + \frac{5}{252}x^7 + \cdots\right]$$

$$+ a_1\left[x - \frac{x^4}{4} - \frac{x^5}{20} + \frac{x^7}{28} + \cdots\right]$$

$$= a_0 y_1(x) + a_1 y_2(x).$$

So, the solution is

$$y(x) = a_0\left[1-\frac{x^3}{3}-\frac{x^4}{12}+\frac{x^6}{18}+\frac{5}{252}x^7+\cdots\right]$$

$$+a_1\left[x-\frac{x^4}{4}-\frac{x^5}{20}+\frac{x^7}{28}+\cdots\right]$$

$$= a_0\, y_1(x) + a_1\, y_2(x) \qquad (*)$$

we can easily show that this is the general solution by showing first that y_1 and y_2 form a fundamental set.

In fact,

$$y_1'(x) = -\frac{3x^2}{3}-\frac{4x^3}{12}+\cdots$$

$$y_2'(x) = 1-\frac{4x^3}{4}+\cdots$$

Therefore, at $x=0$

$$W[y_1,y_2](0) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

This shows $\{y_1, y_2\}$ is a fundamental set and $(*)$ represents the general soln. of

$$y'' + x^2 y' + (x^2+2x)y = 0.$$

Exercise # 2

Find the general Solution of

$y'' - xy' - y = 0, \quad x_0 = 0.$

using power series.

Answer: $$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{\substack{n=0 \\ \text{or} \\ n=2}}^{\infty} n(n-1) a_n x^{n-2}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} \overset{m=n-2}{=\!=} \sum_{\substack{m=-2 \text{ or} \\ m=0}}^{\infty} (m+2)(m+1) a_{m+2} x^m$$

$$xy' = x \sum_{\substack{n=0 \\ \text{or} \\ n=1}}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} n a_n x^n$$

Therefore,
$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - (n+1) a_n \right] x^n = 0$$

Then,
$$a_{n+2} = \frac{\cancel{n+1}}{(n+2)\cancel{(n+1)}} a_n = \frac{a_n}{n+2}, \quad n = 0, 1, 2 \ldots$$

<u>n even:</u>

$k=1$ $\quad n=0: \quad a_2 = \frac{a_0}{2}$

$k=2$ $\quad n=2: \quad a_4 = \frac{a_2}{4} = \frac{a_0}{2 \cdot 4}$

$k=3$ $\quad n=4 \quad a_6 = \frac{a_4}{6} = \frac{a_0}{2 \cdot 4 \cdot 6}$

$\vdots \quad \vdots$

k $\quad n = 2k-2: \quad a_{2k} = \frac{a_{2k-2}}{2k} = \frac{a_0}{2 \cdot 4 \cdot 6 \cdots (2k-2)(2k)}$

$$= \frac{a_0}{\underbrace{2}_{1} \cdot \underbrace{(2)(2)}_{2} \cdot \underbrace{(2)(3)}_{3} \cdots \underbrace{(2)k}_{k}}$$

$$= \frac{a_0}{2^k (1 \cdot 2 \cdot 3 \cdot \ldots \cdot k)}$$

Summarizing:

$$\boxed{a_{2k} = \frac{a_0}{2^k\, k!}}, \quad k = 1, 2, \ldots$$

$\underline{n \text{ odd:}}$ $\quad a_{n+2} = \dfrac{a_n}{n+2}, \quad n = 0, 1, 2 \ldots$

$k=1 \quad n=1: \quad a_3 = \dfrac{a_1}{3}$

$k=2 \quad n=3 \quad a_5 = \dfrac{a_3}{5} = \dfrac{a_1}{3 \cdot 5}$

$k=3 \quad n=5 \quad a_7 = \dfrac{a_5}{7} = \dfrac{a_1}{3.5.7}$

$\vdots \quad \vdots$

Key

$k \quad \boxed{n = 2k-1} \quad a_{2k+1} = \dfrac{a_{2k-1}}{2k+1} = \dfrac{a_1}{3.5.7.\ldots(2k+1)}$

$$= \frac{a_1 \, (2)(4)\ldots(2k)}{(2.3).(4.5).(6.7)\ldots(2k)(2k+1)}$$

$$= \frac{2^k (1.2\ldots k)\, a_1}{(2k+1)!}$$

$$= \frac{2^k k! \, a_1}{(2k+1)!}$$

Summarizing,

$$\boxed{a_{2k+1} = \frac{k! \, 2^k \, a_1}{(2k+1)!},} \quad k = 1, 2, \ldots$$

Finally,

$$y(x) = a_0 + a_1 x + a_0 \sum_{k=1}^{\infty} \frac{1}{2^k k!} x^{2k}$$

$$+ a_1 \sum_{k=1}^{\infty} \frac{2^k k!}{(2k+1)!} x^{2k+1}$$

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{1}{2^k k!} x^{2k}$$

$$+ a_1 \sum_{k=0}^{\infty} \frac{2^k k!}{(2k+1)!} x^{2k+1}$$

$$= a_0 S_0(x) + a_1 S_1(x).$$

Now, we will find out the radius of convergence for $S_1(x)$ and $S_2(x)$

For S_1: $$\lim_{\to\infty} \frac{\frac{1}{2^{k+1}(k+1)!}}{\frac{1}{2^k k!}} \frac{|x|^{2k+2}}{|x|^{2k}} = |x|^2 \lim_{k\to\infty} \frac{2^k k!}{2^{k+1}(k+1)!}$$

$$|x|^2 \lim_{k\to\infty} \frac{1}{2(k+1)} = 0 < 1, \text{ for all } x$$

So radius of convergence $\rho = +\infty$.

For S_2:
$$\lim_{k\to\infty} \frac{2^{k+1}(k+1)!\,(2k+1)!}{(2k+3)!\; 2^k k!} \frac{|x|^{2k+3}}{|x|^{2k+1}}$$

$$|x|^2 \lim_{k\to\infty} \frac{2(k+1)}{(2k+2)(2k+3)} = 0 < 1 \text{ for all } x$$

Therefore, the General Soln:

$$y(x) = a_0 \underbrace{\sum_{k=0}^{\infty} \frac{1}{2^k k!} x^{2k}}_{=y_1(x)} + a_1 \underbrace{\sum_{k=0}^{\infty} \frac{2^k k!}{(2k+1)!} x^{2k+1}}_{=y_2(x)}$$

Converges on $(-\infty, \infty)$ with $\rho = \infty$

On the other hand,

$$y_1'(x) = \left(1 + \frac{1}{2}x^2 + \cdots\right)' = x + \cdots$$

$$y_2'(x) = \left(x + \frac{2}{3!}x^3 + \cdots\right)' = 1 + \cdots$$

Then,

$$y_1'(0) = 0$$

$$y_2'(0) = 1$$

Finally,

$$W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

$\{y_1, y_2\}$ is a fund. set of solutions.

Sect. 5.2 : Problem 19

$$y'' + (x-1)^2 y' + (x^2-1)\,y = 0 \quad (1) \longrightarrow y'' + (x-1)^2 y' + [(x-1)^2 + 2(x-1)]\,y = 0$$

Find two linearly independent solutions.

Change variables : $t = x-1$. Assume $\tilde{y}(t) = \sum_{n=0}^{\infty} a_n t^n$ (2)

At least discuss until here.

$x - 1 = t \Rightarrow x = t+1 \Rightarrow x^2 - 1 = (t+1)^2 - 1 = t^2 + 2t$

Also, $\tilde{y}(t) = y(x(t)) = y(t+1)$

$$\Rightarrow \quad \frac{d\tilde{y}}{dt} = \frac{dy}{dt}\frac{dx}{dt} = \frac{dy}{dt}\,1 = \frac{dy}{dt}.$$

Equ. (1) in terms of t transforms into

$$\boxed{\tilde{y}''(t) + t^2 \tilde{y}' + (t^2 + 2t)\tilde{y} = 0} \qquad (3)$$

Subst. of (2) into (3)

$$\sum_{n=2}^{\infty} n(n-1)\,a_n t^{n-2} + t^2 \sum_{n=1}^{\infty} n a_n t^{n-1} + (t^2 + 2t) \sum_{n=0}^{\infty} a_n t^n = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} (m+2)(m+1)\,a_{m+2} t^m + \sum_{n=1}^{\infty} n a_n t^{n+1} + \sum_{n=0}^{\infty} a_n t^{n+2} + \sum_{n=0}^{\infty} 2 a_n t^{n+1} = 0 \qquad m = n-2$$

$$\Rightarrow \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} t^m + \overset{m=n+1}{\sum_{m=2}^{\infty}} (m-1) a_{m-1} t^m + \overset{m=n+2}{\sum_{m=2}^{\infty}} a_{m-2} t^m +$$

$$+ \overset{m=n+1}{\sum_{m=1}^{\infty}} 2 a_{m-1} t^m = 0$$

$$\Rightarrow \sum_{m=2}^{\infty} \Big[(m+2)(m+1) a_{m+2} + \overbrace{(m-1+2)}^{m+1} a_{m-1} + a_{m-2} \Big] t^m$$

$$+ 2(1) a_2 + \big[(3)(2) a_3 + 2 a_0 \big] t$$

$\therefore$ Next page

$$2a_2 = 0 \Rightarrow \underline{\underline{a_2 = 0}}$$

$$a_3 = \frac{-2a_0}{6} = -\frac{1}{3}a_0$$

interesting because coefficients are coupled!

$$a_{m+2} = \frac{-(m+1)a_{m-1} - a_{m-2}}{(m+2)(m+1)}, \qquad n = 2, 3, \ldots$$

$m=2$: $$a_4 = \frac{-3a_1 - a_0}{(4)(3)} = -\frac{a_1}{4} - \frac{a_0}{12}$$

$m=3$: $$a_5 = \frac{-4a_2 - a_1}{(5)(4)} = -\cancel{\frac{a_2}{5}}^{\,0} - \frac{a_1}{20} = -\frac{a_1}{20}$$

$m=4$: $$a_6 = \frac{-5a_3 - a_2}{(6)(5)} = -\frac{a_3}{6} - \cancel{\frac{a_2}{30}}^{\,0} = -\frac{a_3}{6} = \frac{a_0}{18}$$

$m=5$: $$a_7 = \frac{-6a_4 - a_3}{(7)(6)} = -\frac{a_4}{7} - \frac{a_3}{42} = \frac{a_1}{28} - \frac{a_0}{84} + \frac{a_0}{126}$$

$$= \frac{a_1}{28} - \frac{a_0}{252}$$

$\therefore$

$$y(t) = a_0\left[1 - \frac{t^3}{3} - \frac{t^4}{12} + \frac{t^6}{18} - \frac{t^7}{252} + \cdots\right]$$

$$+ a_1\left[t - \frac{t^4}{4} - \frac{t^5}{20} + \frac{t^7}{28} + \cdots\right] \checkmark$$

Finally,

$$y(x) = a_0\left[1 - \frac{(x-1)^3}{3} - \frac{(x-1)^4}{12} + \frac{(x-1)^6}{18} - \frac{(x-1)^7}{252} + \cdots\right]$$

$$+ a_1\left[x-1 - \frac{(x-1)^4}{4} - \frac{(x-1)^5}{20} + \frac{(x-1)^7}{28} + \cdots\right]$$

Two important cases:

A) $P(x)$, $Q(x)$, and $R(x)$ are polynomials with no common factors $(x-x_0)$.

B) $$p(x) = \frac{Q(x)}{P(x)}, \quad q(x) = \frac{R(x)}{P(x)}$$

are analytic functions at $x=x_0$.

CASE A: Polynomials

Def. i) If x_0 is such that $P(x_0) \neq 0$ then, $x = x_0$ is called an ordinary point of

$$P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0 \qquad (*)$$

ii) If $P(x_0) = 0$ and at least $Q(x_0) \neq 0$ or $R(x_0) \neq 0$, then $x = x_0$ is called a singular point of $(*)$.

Determine the Singular points for

5.2

\# 10) $(4-x^2)y''+2y=0$

Here, $P(x)=4-x^2 \Rightarrow x^2=4$ or $x=\pm 2$

are Singular points. Thus, $x_0=0$ is an

ordinary point and we can try to find a Soln. as

$$y(x)=\sum_{n=0}^{\infty} a_n x^n$$

5.3

\# 7) $(1+x^3)y''+4xy'+y=0$

Here, $P(x)=1+x^3$ the $x=-1$ is a Singular

point. We can also look for a Soln as

$$y(x)=\sum_{n=0}^{\infty} a_n x^n$$

5.4

\# 14) $4x^2y''+8xy'+17y=0, \quad x>0$

$x_0=0$ is a singular point, because

$P(x)=4x^2\big|_{x_0=0}=0$.

As a consequence, <u>we do not expect a power</u>

<u>Series Solution</u> as

$$y(x)=\sum_{n=0}^{\infty} a_n x^n$$

Soln. discussed in Sect 5.4

Procedure: If x_0 is an ordinary point,

① Propose as Solution a power Series

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

② Subst. into the equation (*)

and find

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 y_1(x) + a_1 y_2(x).$$

where a_0 and a_1 are arbitrary constants and $y_1(x)$ and $y_2(x)$ form a fundamental set of of Solutions of (*).

③ Determine the radius of Convergence ρ of the power series $y_1(x)$ and $y_2(x)$.

This Can be done if the general term of the power Series is known.

If not Compute the distance from x_0 to the nearest zero of $P(x)$ (including Complex roots).

The radius of convergence of y_1 and y_2 is at least as large as the distance from x_0 to the nearest zero of $P(x)$.

Why?

Two Important theorems supporting this procedure

Weaker Version of Theorem 5.3.1

Theorem. i) If $P(x)$, $Q(x)$, and $R(x)$ are polynomials and $(x-x_0)$ is not a common factor to all of them.

ii) If $p(x) = \frac{Q(x)}{P(x)}$, $q(x) = \frac{R(x)}{P(x)}$, and $P(x_0) \neq 0$.

It means x_0 is an ordinary point of (*).

Then,

A) The general solution of (*) is given by

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 y_1(x) + a_1 y_2(x)$$

where a_0, a_1 are arbitrary and y_1 and y_2 are two power series solutions that converges at $x = x_0$.

B) The solutions y_1 and y_2 form a fundamental set.

C) The radius of convergence for each of the series solutions y_1 and y_2 is at least as large as the minimum of the radius of convergence of the series for $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$.

<u>Theorem</u> i) If $P(x)$ and $Q(x)$ are polynomials without common factors

ii) $P(x_0) \neq 0$

Then,

A) $\dfrac{Q(x)}{P(x)}$ has a convergent power series expansion about $x = x_0$.

B) The radius of convergence of this power series about $x = x_0$ is the distance from x_0 to the nearest zero of $P(x)$.

Back to problems

5.2 # 10) $(4-x^2)y''+2y=0 \Rightarrow x=\pm 2$ Sing. points

According to thm 3.5.1,

$y(x) = a_0 y_1(x) + a_1 y_2(x)$ is

where y_1 and y_2 are power series Soln. (convergent)

about any $x_0 \neq \pm 2$. In particular,

y_1 and y_2 can be power series Solution

about $x_0 = 0$.

Also, the radius of convergence of y_1 and y_2

is at least $\rho = 2$.

5.3 #7) $(1+x^3)y''+4xy'+y=0$

$x=-1$ only Sing. point

Using thm 3.5.1

the general soln. of this equation can be

represented as

a) $y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 \underbrace{y_1(x)}_{\text{power series}} + a_2 \underbrace{y_2(x)}_{\text{power series}}$. at $x_0 = 0$

or

b) $y(x) = a_0 \tilde{y}_1(x) + a_2 \tilde{y}_2(x)$

where $\tilde{y}_1 = \sum_{n=0}^{\infty} b_n (x-2)^n$, $\tilde{y}_2 = \sum_{n=0}^{\infty} c_n (x-2)^n$

and the radius of convergence $\rho_{a,b}$ of y_1 and y_2 and $\tilde{y}_1, \tilde{y}_2$
will be at least as large as
the distance from $x_0 = 0, 2$ to the roots of

$x^3 + 1 = 0$ or

$$x = (-1)^{1/3} = \left(e^{i(\pi + 2m\pi)}\right)^{1/3} \quad m = 0, 1, 2$$

$$m=0: \ x_1 = e^{i\pi/3} = \cos \pi/3 + i \sin \pi/3 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$m=1: \ x_2 = e^{i(\pi/3 + 2\pi/3)} = e^{i\pi} = -1$$

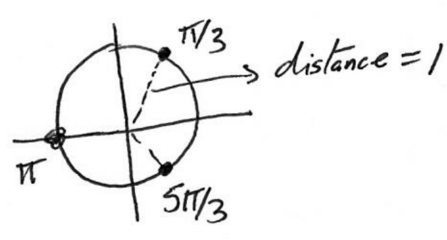
$$m=2: \ x_3 = e^{i\left(\pi/3 + \frac{4\pi}{3}\right)} = e^{i(5\pi/3)}$$

$$= \cos(5\pi/3) + i\sin(5\pi/3) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Then,

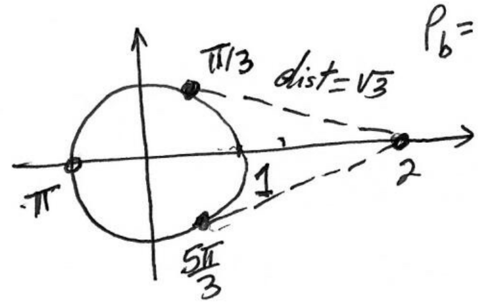
a) For y_1 and y_2 power series about $x_0 = 0$.

$\rho_a \geq 1$, because



b) for $\tilde{y}_1$ and $\tilde{y}_2$ P.S. about $x_0 = 2$

$\rho_b \geq \sqrt{3}$, because



$$\rho_b = \sqrt{\left(2-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\left(\frac{3}{2}\right)^2 + \frac{3}{4}} = \sqrt{\frac{12}{4}} = \sqrt{3}.$$

Analytic Functions

Consider

$\sin x$, $\cos x$ and e^x.

They have power series expansions.
given by

$$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$g(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!}$$

$$h(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Where do these power series come from?

Notice that

$$f'(0) = \cos x \big|_{x=0} = 1$$

$$\frac{f''(0)}{2} = \frac{(\cos x)'}{2}\Big|_{x=0} = \frac{-\sin x}{2}\Big|_{x=0} = -\frac{\sin 0}{2} = 0$$

$$\frac{f'''(0)}{3!} = \frac{(-\sin x)'}{3!}\Big|_{x=0} = \frac{-\cos x}{3!}\Big|_{x=0} = -\frac{1}{3!}$$

$$\frac{f^{(2n+1)}(0)}{(2n+1)!} = \frac{(-1)^n}{(2n+1)!}, \quad \frac{f^{(2n)}(0)}{(2n)!} = 0$$

Therefore, the Taylor Series expansion of $\sin x$.
around $x=0$.

$$\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} x^n = \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} x^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Similarly, for $h(x) = e^x$

$$h'(x) = e^x, \quad h'(0) = e^0 = 1$$

$$h''(x) = e^x, \quad h''(0) = e^0 = 1, \ldots, \quad h^{(n)}(x) = e^x\Big|_{x=0} = 1$$

Then, Taylor Series expansion of $h(x) = e^x$
around $x_0 = 0$.

$$e^x = \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

It is important to know where (in what interval) they converge.

<u>Ratio Test</u>:

<u>Sinx</u>: $\lim\limits_{n\to\infty} \dfrac{|a_{2n+1}||x|^{2n+1}}{|a_{2n-1}||x|^{2n-1}} = \left|\dfrac{(-1)^{2n+1}}{(2n+1)!}\right| |x^{2n+1}| \Big/ \dfrac{(-1)^{2n-1}}{(2n-1)!} |x^{2n-1}|$

$= \lim\limits_{n\to\infty} \dfrac{(2n-1)!\,|x|^{2n+1}}{(2n+1)!\,|x|^{2n-1}} =$

$= \lim\limits_{n\to\infty} \dfrac{|x|^2}{(2n+1)(2n)} = 0 < 1$ for all x.

It means the Taylor Series of $\sin x$ converges in $(-\infty, \infty)$

Similarly

e^x: $\lim\limits_{n\to\infty} \dfrac{|a_{n+1}||x|^{n+1}}{|a_n||x|^n} = \lim\limits_{n\to\infty} \dfrac{|x|^{n+1}/(n+1)!}{|x|^n/n!}$

$= \lim\limits_{n\to\infty} \dfrac{n!\,|x|^{n+1}}{(n+1)!\,|x|^n} = \lim\limits_{n\to\infty} \dfrac{|x|}{n+1} = 0 < 1$

for all x. $(-\infty, \infty)$.

ratio of convergence is $\rho = \infty$.

Definition:

A function $f(x)$ is Analytic at x_0 if it has a Taylor Series expansion that converges to $f(x)$ in some interval $(x_0-\rho, x_0+\rho)$ about x_0.

Example: $\sin x$, $\cos x$ and e^x are analytic around $x_0=0$ and their radius of convergence is $\rho=\infty$.

Consider now,

5.3 #12) $e^x y'' + xy = 0$ (**)

$P(x) = e^x$ not a polynomial

$Q(x) = 0, \quad R(x) = x.$

What can be said abut power series solutions?

First, we need to extend the notion of ordinary and singular points for equations as (**) where $P(x)$, $Q(x)$, or $R(x)$ are not polynomials.

For this, we also need the definition of analytic functions.

<u>Def.</u>- Consider

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

and $y'' + p(x)y' + q(x)y = 0$, where $p = \frac{Q}{P}$, $q = \frac{R}{P}$.

a) The functions <u>p, q are analytic at $x = x_0$</u> if they have Taylor series expansions that converge to them in some interval about $x = x_0$.

$$p(x) = p_0 + p_1(x - x_0) + \cdots + p_n(x - x_0)^n + \cdots$$

$$q(x) = q_0 + q_1(x - x_0) + \cdots + q_n(x - x_0)^n + \cdots$$

b) If the functions $p(x)$ and $q(x)$ are analytic at $x = x_0$, the point $x = x_0$ is an ordinary point. otherwise, it is a singular point

Thm 5.3.1

i) If x_0 is an ordinary point of $(*)$, which is equivalent to say $p = \frac{Q}{P}$ and $q = \frac{R}{P}$ are analytic at x_0

Then,

A) The general soln. of $(*)$ is given by

$$y(x) = a_0\, y_1(x) + a_1\, y_2(x)$$

where a_0, a_1 arbitrary and y_1, y_2 are two power series solution that converges at $x = x_0$.

B) The solns. y_1 and y_2 form a fundamental set.

C) The radius of convergence for y_1 and y_2 is at least as large as the minimum of the radius of convergence of the series p and q.

5.3 #12) $e^x y'' + xy = 0.$ (Very good!)

$$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} a_n x^n = 0.$$

or

$$\sum_{2} n(n-1) a_n x^{n-2} + \sum_{n=2} n(n-1) a_n x^{n-1} + \sum_{n=2} \frac{n(n-1)}{2} a_n x^n + \sum_{2} \frac{n(n-1)}{6} a_n x^{n+1} +$$

$$+ \sum_{2} \frac{n(n-1)}{\cancel{12}} a_n x^{n+2} + \sum_{2} \frac{n(n-1)}{120} a_n x^{n+3} + \sum_{n=0} a_n x^{n+1} = 0$$

or

$$\sum_{0} (n+2)(n+1) a_{n+2} x^n + \sum_{1} (n+1) n a_{n+1} x^n + \sum_{2} \frac{n(n-1)}{2} a_n x^n + \sum_{3} \frac{(n-1)(n-2)}{6} a_{n-1} x^n$$

$$+ \sum_{4} \frac{(n-2)(n-3)}{12} a_{n-2} x^n + \sum_{5} \frac{(n-3)(n-4)}{120} a_{n-3} x^n + \sum_{n=1} a_{n-1} x^n = 0.$$

x^0: $2(1) a_2 = 0 \Rightarrow \boxed{a_2 = 0}$

x^1: $3(2) a_3 + 2(1) \cancel{a_2}^{0} + a_0 = 0$
$\Rightarrow \boxed{a_3 = -\frac{a_0}{6}}$

x^2: $4(3) a_4 + 3(2) a_3 + \frac{2(1)}{2} \cancel{a_2}^{0} + a_1 = 0$
$12 a_4 + 6a_3 + a_1 = 0 \Rightarrow a_4 = \frac{1}{12}(-a_1 - 6a_3)$

$\Rightarrow \boxed{a_4 = \frac{1}{12}(a_0 - a_1)}$

$$x^3: \quad 5(4)a_5 + 4(3)a_4 + \frac{3(2)}{2}a_3 + \left(\frac{2(1)}{6}+1\right)a_2$$

$$\Rightarrow a_5 = \frac{-1}{20}\left[12a_4 + 3a_3 + \frac{4}{3}\cancel{a_2}^{\,0}\right]$$

$$\Rightarrow a_5 = -\frac{1}{20}\left[a_0 - a_1 - \frac{a_0}{2}\right] = -\frac{1}{20}\left[\frac{a_0}{2} - a_1\right]$$

so $\boxed{a_5 = -\frac{1}{20}\left[\frac{a_0}{2} - a_1\right]}$

Therefore,

$$y(x) = a_0 + a_1 x - \frac{a_0}{6}x^3 + \frac{1}{12}(a_0 - a_1)x^4 - \frac{1}{20}\left(\frac{a_0}{2} - a_1\right)x^5 + \dots$$

or

$$y(x) = a_0\left[1 - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{40} + \dots\right] + $$

$$+ a_1\left[x - \frac{x^4}{12} + \frac{x^5}{20} \quad \dots\right]$$

$$x^4: \quad 6(5)a_6 + 5(4)a_5 + \frac{4(3)}{2}a_4 + \left[\frac{3(2)}{6} + 1\right]a_3 +$$

$$+ \frac{2(1)}{12}\cancel{a_2}^{\,0} = 0.$$

$$a_6 = \frac{-1}{30}\left[20a_5 + 6a_4 + 2a_3\right] = -\frac{1}{30}\left[a_1 - \cancel{\frac{a_0}{2}} + \cancel{\frac{1}{2}a_0} - \frac{1}{2}a_1 - \frac{a_0}{3}\right] = -\frac{a_1}{60} + \frac{a_0}{90}$$

$$\Rightarrow \boxed{a_6 = -\frac{a_1}{60} + \frac{a_0}{90}}$$

$$\boxed{a_6 = -\frac{a_1}{60} + \frac{a_0}{90}}$$

Therefore,

$$y(x) = a_0\left[1 - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{40} + \frac{x^6}{90} + \cdots\right]$$

$$+ a_1\left[x - \frac{x^4}{12} + \frac{x^5}{20} - \frac{x^6}{60} + \cdots\right]$$