Arc length

$\mathcal{C}: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$a \le t \le b$

$$\int_{\mathcal{C}} ds = \int_a^b \|\vec{r}'(t)\| \, dt$$

$$= \int_a^b \sqrt{x'^2(t) + y'^2(t) + z'^2(t)} \, dt$$

Surface Area

$S: \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

$u, v \in D_{u,v}$

$$\iint_S ds = \iint_{D_{u,v}} \| \vec{r}_u \times \vec{r}_v \| \, du\,dv$$

Real functions (Scalar functions) $f(x,y,z)$

Line Integrals

$\mathcal{C}: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$a \le t \le b$

$$\int_{\mathcal{C}} f(x,y,z)\, ds = \int_a^b f\big(\underbrace{x(t),y(t),z(t)}_{=\vec{r}(t)}\big) \|\vec{r}\,'(t)\|\, dt$$

$$= \int_a^b f\big(x(t),y(t),z(t)\big) \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\, dt$$

Surface Integral

$S: \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

u,v in $D_{u,v}$

$$\iint_S f(x,y,z)\, dS = \iint_{D_{u,v}} f(\vec{r}(u,v))\, \|\underbrace{\vec{r}_u \times \vec{r}_v}_{=\vec{n}(u,v)}\|\, du\, dv$$

Vector Functions

Line Integral

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_{\mathcal{C}} \vec{F} \cdot \vec{T}\, ds$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\cancel{\|\vec{r}\,'(t)\|}} \cancel{\|\vec{r}\,'(t)\|}\, dt$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt.$$

Surface Integral

$$\iint_S \vec{F} \cdot d\vec{s} = \iint_S \vec{F} \cdot \hat{n}\, ds$$

$$= \iint_{D_{u,v}} \vec{F}(\vec{r}(u,v)) \cdot \underbrace{\left(\frac{\vec{r}_u \times \vec{r}_v}{\cancel{\|\vec{r}_u \times \vec{r}_v\|}}\right)}_{= \hat{n}(u,v)} \cancel{\|\vec{r}_u \times \vec{r}_v\|}\, du\, dv$$

$$= \iint_{D_{u,v}} \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv.$$

Parametric Surfaces
Area and Surface Integrals

1) Parametrization.

$$\vec{r} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$(u,v) \longrightarrow \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle .$$

Examples:

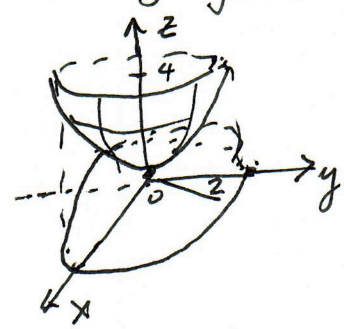
a) Paraboloid:

$z = x^2 + y^2, \quad 0 \leq z \leq 4 .$

$x = x(r,\theta) = r\cos\theta$

$y = y(r,\theta) = r\sin\theta$

$$S: \begin{cases} \vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2 \rangle \\ 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 2 . \end{cases}$$



b) Half Cone: $z^2 = (x^2 + y^2), \quad z \geq 0$

Spherical Coordinates: $\phi = \pi/4, \quad \rho \geq 0, \quad 0 \leq \theta \leq 2\pi .$

$$S: \begin{cases} \vec{r}(\rho,\theta) = \langle \rho \sin\pi/4 \cos\theta, \rho\sin\pi/4 \sin\theta, \rho\cos\pi/4 \rangle \\ \quad = \langle \frac{\sqrt{2}\rho}{2}\cos\theta, \frac{\sqrt{2}\rho}{2}\sin\theta, \frac{\sqrt{2}}{2}\rho \rangle \\ \rho \geq 0, \quad 0 \leq \theta \leq 2\pi . \end{cases}$$

Ask for:

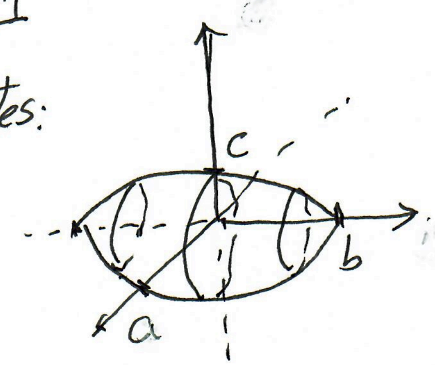
c) Half cone using Cylindrical Coordinates.

$$S: \begin{cases} x = x(r,\theta) = r\cos\theta, \\ y = y(r,\theta) = r\sin\theta, \\ z = z(r,\theta) = r, \end{cases} \quad 0 \le \theta \le 2\pi, \quad r \geq 0.$$

$$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r \rangle$$

d) Ellipsoid with semi-axes: a, b, c.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



Using spherical Coordinates:

$$x = a\sin\phi\cos\theta$$

$$y = b\sin\phi\sin\theta$$

$$z = c\cos\phi. \qquad 0 \le \phi \le \pi, \qquad 0 \le \theta \le 2\pi$$

Since

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{\cancel{a^2}\sin^2\phi\cos^2\theta}{\cancel{a^2}} + \frac{\cancel{b^2}\sin^2\phi\sin^2\theta}{\cancel{b^2}} + \frac{\cancel{c^2}\cos^2\phi}{\cancel{c^2}}$$

$$= \sin^2\phi\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} + \cos^2\phi = 1$$

Find the surface area of the cone

$$\boxed{z^2 = x^2 + y^2,} \qquad 0 \le z \le 1.$$

Answ: Using spherical coords, we can show $\phi = \pi/4$.

and

$$\vec{r}(r,\theta) = \begin{cases} x(\rho,\theta) = \rho \sin \pi/4 \cos\theta = \frac{\sqrt{2}}{2}\rho\cos\theta, \\ y(\rho,\theta) = \rho \sin \pi/4 \sin\theta = \frac{\sqrt{2}}{2}\rho\sin\theta, \\ z(\rho,\theta) = \rho\cos\pi/4 = \frac{\sqrt{2}}{2}\rho, \end{cases}$$

$0 \le \theta \le 2\pi$

$0 \le z \le 1 \Rightarrow 0 \le \rho \le \frac{2}{\sqrt{2}}$

Then

$$\vec{n} = \vec{r}_\rho \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sqrt{2}/2\cos\theta & \sqrt{2}/2\sin\theta & \sqrt{2}/2 \\ -\sqrt{2}/2\,\rho\sin\theta & \sqrt{2}/2\,\rho\cos\theta & 0 \end{vmatrix}$$

$$= \left\langle -\frac{1}{2}\rho\cos\theta, -\frac{1}{2}\rho\sin\theta, \frac{1}{2}\rho\cos^2\theta + \frac{1}{2}\rho\sin^2\theta \right\rangle$$

$$= \frac{1}{2}\rho\left\langle -\cos\theta, -\sin\theta, 1\right\rangle$$

Thus,

$$S_{(\text{surface area})} = \frac{1}{2}\int_{\substack{0\\ \theta}}^{2\pi}\int_{\substack{0\\ \rho}}^{2/\sqrt{2}} \rho\sqrt{\cos^2\theta + \sin^2\theta + 1}\,d\rho\,d\theta$$

$$= \frac{\sqrt{2}}{\cancel{2}}(\cancel{2}\pi)\left.\frac{\rho^2}{\cancel{2}}\right|_0^{2/\sqrt{2}} = \frac{\sqrt{2}}{\cancel{2}}\pi\frac{4}{\cancel{2}} = \underline{\underline{\sqrt{2}\,\pi}}$$

Problems:

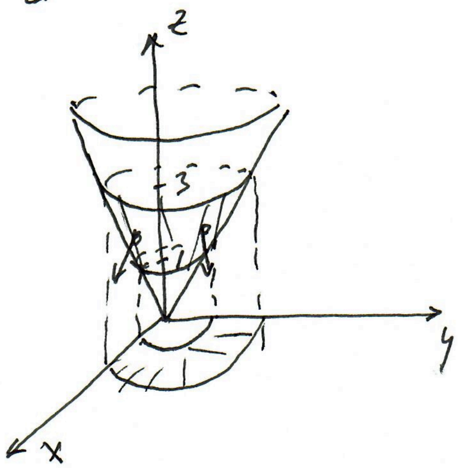
16.7 #24 Evaluate the Surface integral

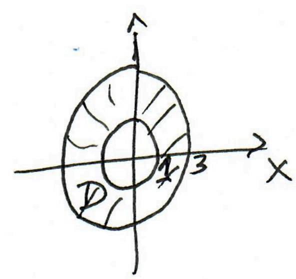
$$\iint_S \vec{F} \cdot d\vec{s}$$

where $\vec{F}(x,y,z) = \langle -x, -y, z^3 \rangle$

S: part of cone between planes $z=1$, $z=3$. with downward orientation.

Cone: $z = \sqrt{x^2+y^2}$





$$\iint_S \vec{F} \cdot d\vec{s} = \iint_S \vec{F} \cdot \hat{n}\, ds =$$

$S = \vec{r}(x,y) = \langle x, y, \sqrt{x^2+y^2} \rangle$,

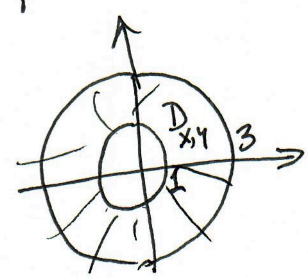
D_{xy}

$1 \leq x^2+y^2 \leq 9$

$$\hat{n}(x,y,z) = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{x}{\sqrt{x^2+y^2}} \\ 0 & 1 & \frac{y}{\sqrt{x^2+y^2}} \end{vmatrix} = \left\langle \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1 \right\rangle$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{D_{x,y}} \vec{F}(x,y,z) \cdot \underbrace{\vec{n}(x,y,z)}_{= \boxed{-\hat{n}(x,y,z) \text{ downward orient.}}} \overset{= dx\,dy}{dA}$$

$$= \iint_{D_{xy}} \langle -x, -y, z^3 \rangle \cdot \left\langle \frac{+x}{r}, \frac{+y}{r}, -1 \right\rangle dx\,dy$$



$$= \iint_{D_{xy}} \left(-\frac{x^2+y^2}{r} - z^3 \right) dx\,dy$$

Cylind. coords

$$= -\int_1^3 \int_0^{2\pi} (r + r^3)\, r\, dr\, d\theta = -\int_1^3 \int_0^{2\pi} (r^2 + r^4)\, dr\, d\theta$$

$$= -2\pi \left[\frac{r^3}{3} + \frac{r^5}{5} \right]\Bigg|_1^3 = -2\pi \left(9 + \frac{243}{5} - \frac{1}{3} - \frac{1}{5} \right)$$

$$= -\frac{1712}{15}\pi.$$

<u>Alternative using Spherical Coordinates for parametization of cone</u>

$$\vec{r} = \vec{r}(\rho,\theta) = \langle \overset{x(\rho,\theta)}{\rho \cos \pi/4 \cos\theta}, \overset{y(\rho,\theta)}{\rho \sin\pi/4 \sin\theta}, \overset{z(\rho,\theta)}{\rho \cos\pi/4} \rangle$$

or $\boxed{\vec{r}(\rho,\theta) = \langle \frac{\sqrt{2}}{2}\rho\cos\theta, \frac{\sqrt{2}}{2}\rho\sin\theta, \frac{\sqrt{2}}{2}\rho \rangle}$

Since $1 \le z \le 3 \Rightarrow 1 \le \frac{\sqrt{2}}{2}\rho \le 3$

or $\frac{2}{\sqrt{2}} \le \rho \le \frac{6}{\sqrt{2}}$

or $\sqrt{2} \le \rho \le 3\sqrt{2}$

So $D_{\rho,\theta} = \{(\rho,\theta) : \sqrt{2} \le \rho \le 3\sqrt{2},\ 0 \le \theta \le 2\pi\}$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{D_{\rho,\theta}} \vec{F}(\vec{r}(\rho,\theta)) \cdot \vec{n}(\rho,\theta)\, d\rho\, d\theta$$

Now, $\boxed{\hat{n}(\rho,\theta) = \frac{1}{2}\langle -\rho\cos\theta, -\rho\sin\theta, \rho \rangle}$

Recall that $\boxed{\vec{F}(x,y) = \langle -x, -y, z^3 \rangle}$.

Then,

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_{\sqrt{2}}^{3\sqrt{2}} \left\langle -\frac{\sqrt{2}}{2}\rho\cos\theta, -\frac{\sqrt{2}}{2}\rho\sin\theta, \frac{2\sqrt{2}}{8}\rho^3 \right\rangle \cdot$$

$$\left\langle -\frac{1}{2}\rho\cos\theta, -\frac{1}{2}\rho\sin\theta, \frac{1}{2}\rho \right\rangle d\rho\, d\theta$$

$$= \int_0^{2\pi}\int_{\sqrt{2}}^{3\sqrt{2}} \Big[\underbrace{\frac{\sqrt{2}}{4}\rho^2\cos^2\theta + \frac{\sqrt{2}}{4}\rho^2\sin^2\theta}_{\sqrt{2}/4\,\rho^2} + \frac{\sqrt{2}}{8}\rho^4 \Big]\, d\rho\, d\theta$$

$$= \int_0^{2\pi} \frac{\sqrt{2}}{4}\frac{\rho^3}{3} + \frac{\sqrt{2}}{8}\frac{\rho^5}{5}\Big|_{\sqrt{2}}^{3\sqrt{2}}\, d\theta$$

$$= \int_0^{2\pi} \Big[\frac{\sqrt{2}}{4}\,9(2\sqrt{2}) + \frac{\sqrt{2}}{8}\,\frac{243(4\sqrt{2})}{5} - \Big(\frac{\sqrt{2}}{4}\Big(\frac{2\sqrt{2}}{3}\Big)\Big) - \frac{\sqrt{2}}{8}\Big(\frac{4\sqrt{2}}{5}\Big)\Big]\, d\theta$$

$$= \int_0^{2\pi} \Big[9 + \frac{243}{5} - \frac{1}{3} - \frac{1}{5}\Big]\, d\theta$$

$$= 2\pi\,[\quad] = \frac{1712}{15}\pi$$

because downward orientation $\boxed{-\frac{1712}{15}\pi.}$

(A) Normal Vector $\vec{n}$, (B) Surface area and (C) Surface integrals in Cartesian Coords.

If $S: \vec{r}(x,y) = \langle x, y, f(x,y) \rangle$

(x,y) in $D_{x,y}$.

A) Then $\boxed{\vec{n}(x,y) = \vec{r}_x(x,y) \times \vec{r}_y(x,y)}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = \boxed{\left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle}$$

Example half cone between $z=1$, $z=3$.

$z = \sqrt{x^2+y^2}$, $\quad D_{xy} = \{ (x,y) : 1 \le x^2+y^2 \le 9$

parametrization: $\vec{r}(x,y) = \langle x, y, \sqrt{x^2+y^2} \rangle$

$$\boxed{\vec{n}(x,y) = \left\langle \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1 \right\rangle}$$

B) Surface integral of $f(x,y,z)$ on S.

$$\boxed{\iint_S f(x,y,z)\, dS = \iint_{D_{x,y}} f(\vec{r}(x,y)) \|\vec{n}(x,y)\| \, dx\, dy}$$

C) Surface integral for a Vector function on S.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n}\, dS$$

$$= \iint_{D_{x,y}} \vec{F}(\vec{r}(x,y)) \cdot \frac{\vec{r}_x \times \vec{r}_y}{\cancel{\|\vec{r}_x \times \vec{r}_y\|}} \cancel{\|\vec{r}_x \times \vec{r}_y\|}\, dx\, dy.$$

Show computation over half-cone.