II Partial Derivatives

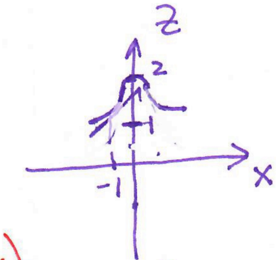
Before formal definition. $\longrightarrow$ show MAPle worksheet

Then, Consider $f(x,y) = e^{-x^2} + e^{-4y^2}$

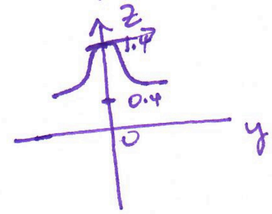
Derivatives at $\boxed{(-1,0)}$

Consider

i) $g(x) \equiv f(x,0) = e^{-x^2} + 1$ $(y=0)$



ii) $h(y) \equiv f(-1,y) = e^{-1} + e^{-4y^2}$ $(x=-1)$



Now,

$$g'(x) = -2x\, e^{-x^2}$$

$$g'(-1) = 2e^{-1} \quad \text{slope.}$$

$$h'(y) = -8y\, e^{-4y^2}$$

$$h'(0) = 0, \quad \text{slope}$$

In general for any y fixed

$$g(x) = f(x, y_{fixed})$$

$$g'(x) = \frac{d}{dx}\left[f(x, y_{fixed}) \right] = -2x\, e^{-x^2} = f_x(x,y) = \frac{\partial f}{\partial x}(x,y)$$

For any x fixed, $h(y) = f(x_{fixed}, y)$

$$h'(y) = \frac{d}{dy}\left[f(x_{fixed}, y) \right] = -8y\, e^{-4y^2} = f_y(x,y) = \frac{\partial f}{\partial y}(x,y).$$

<u>Example:</u>

$$f(x,y) = e^{-x^2} + e^{-4y^2}$$

We want to analize 1-D derivatives at the point $(-1,0)$. First, we define

i) $g(x) = f(x,0) = e^{-x^2} + 1$

ii) $h(y) = f(-1,y) = e^{-1} + e^{-4y^2}$

~~g(x) is~~ The graph of $g(x)$ is the curve ~~at~~ the intersection of the plane $y=0$ with the Surface S corresponding to the given fn. $f(x,y)$.

The graph of $h(y)$ is the curve at the intersection of the plane $x=-1$ with the Surface S corresponding to the given fn. $f(x,y)$.

Show MAPLE worksheet.

We can compute $g'(-1)$ and $h'(0)$.

In fact, $\cancel{g'(-1) = -2x\, e^{-x^2}\Big|_{x=-1} = 2e^{-1}}$

~~and~~ $\cancel{h'(0) =}$

$g'(x) = -2x\,e^{-x^2}$ everywhere $\Rightarrow$ It is also written as $f_x(x,y) \Rightarrow g'(-1) = 2e^{-1}$

$h'(y) = -8y\,e^{-4y^2}$ everywhere $= f_y(x,y) \Rightarrow h'(0) = 0.$

$f_x(x,y)$ is a new function called partial derivative of $f(x,y)$ w.r.t. x.

$f_y(x,y)$ is a new function " " " $f(x,y)$ w.r.t. y.

or derivative in the direction of x at the point $(-1,0)$.

or derivative in the direction of y at the point $(-1,0)$

From the above results,

$$f_x(-1,0) = g'(-1) = 2e^{-1}$$

$$f_y(-1,0) = h'(0) = 0$$

these values are the ~~slopes~~ numerical values of the slopes of the tangent lines to the graph of $g(x)$ at $x=-1$ and to the graph of $h(y)$ at $y=0$, respectively.

Show MAPLE Graphs. with tangent vectors.

It is evident from the graphs that if we choose another plane the curve ~~of~~ the intersection and the slope will change.

<u>Def.</u> – Let $f(x,y)$ be a real-valued function of two variables.

i) The P.D. w.r.t. x is the fn.

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

ii) the P.D. w.r.t. y is the fn.

$$f_y(x,y) = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

provided that these two limit exists.

~~Example~~ <u>COMPUTATION</u>

~~For~~ The computation of $f_x(x,y)$ and $f_y(x,y)$ when they exist can be done directly without defining $g(x)$ and $h(y)$. ~~In fact,~~ For example, $f_x(x,y)$ can be obtained directly from $f(x,y)$ Considering y constant and taking the derivative w.r.t. x.

In our example,

$$f(x,y) = e^{-x^2} + e^{-4y^2}$$

$$\Rightarrow f_x(x,y) = -2x e^{-x^2} + 0.$$

Similarly, $f_y(x,y) = 0 - 8y e^{-4y^2}$.

II

$$f(x,y) = xy^2 - x^3y$$

Use definition to get $f_x(x,y)$

without the definition: $f_x(x,y) = y^2 - 3x^2y.$

Now, using definition:

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)y^2 - (x+h)^3y - xy^2 + x^3y}{h}.$$

Now, $(x+h)^3 = (x+h)^2(x+h) = (x^2+2hx+h^2)(x+h)$

$$= x^3 + hx^2 + 2hx^2 + 2h^2x + h^2x + h^3$$

$$= x^3 + 3hx^2 + 3h^2x + h^3$$

$$\therefore \lim_{h\to 0} \frac{\cancel{xy^2} + hy^2 - \cancel{x^3y} - 3hx^2y - 3h^2xy - h^3y - \cancel{xy^2} + \cancel{x^3y}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(y^2 - 3x^2y - 3hxy - h^2y)}{\cancel{h}} = y^2 - 3x^2y$$

14.3

#76) Show that $u(x,y) = \ln\sqrt{x^2+y^2} = \ln(x^2+y^2)^{1/2}$

is a solution of

$$u_{xx} + u_{yy} = 0$$

<u>answ:</u>

$$u_x = \frac{1}{(x^2+y^2)^{1/2}} \cdot \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2} \cdot \cancel{2}x$$

$$= \frac{x}{x^2+y^2} = x(x^2+y^2)^{-1}$$

$$u_{xx} = (x^2+y^2)^{-1} - x(x^2+y^2)^{-2} \cdot 2x$$

$$= \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}$$

Similarly,

$$u_{yy} = \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}$$

Thus,

$$u_{xx} + u_{yy} = \frac{2}{x^2+y^2} - \frac{2\cancel{(x^2+y^2)}}{(x^2+y^2)^2} =$$

$$= \frac{2}{x^2+y^2} - \frac{2}{x^2+y^2} = 0$$