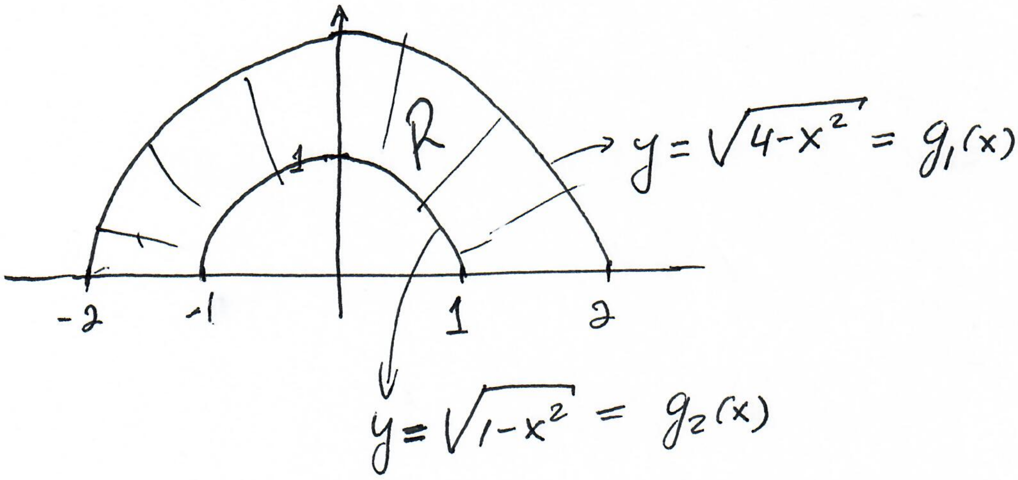
15.3 Double Integrals in Polar Coordinates

Example 1.

Compute
$$I = \iint_R (3x + 4y^2)\, dA$$

where R is in the upper half-plane bounded by the curves
$$x^2+y^2=1 \quad \text{and} \quad x^2+y^2=4$$



Then,
$$I = \iint_R (3x+4y^2)dA = \int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} (3x+4y^2)\,dy\,dx \; -$$
$$- \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (3x+4y^2)\,dy\,dx$$

Then,

$$I = \int_{-2}^{2} \left(3xy + \frac{4}{3} y^3 \Bigg|_{y=0}^{y=\sqrt{4-x^2}} \right) dx -$$

$$- \int_{-1}^{1} \left(\left(3xy + \frac{4}{3} y^3\right) \Bigg|_{y=0}^{y=\sqrt{1-x^2}} \right) dx$$

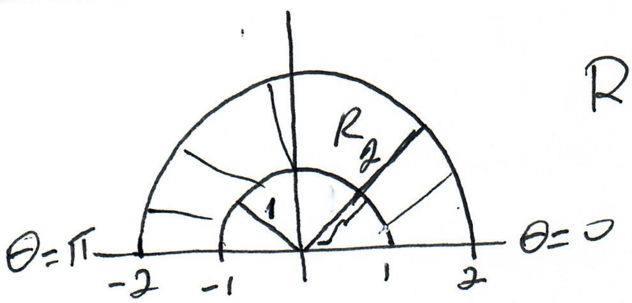
$$= \int_{-2}^{2} \left[3x\sqrt{4-x^2} + \frac{4}{3}(4-x^2)^{3/2} \right] dx -$$

$$- \int_{-1}^{1} 3x\sqrt{1-x^2} + \frac{4}{3}(1-x^2)^{3/2} dx$$

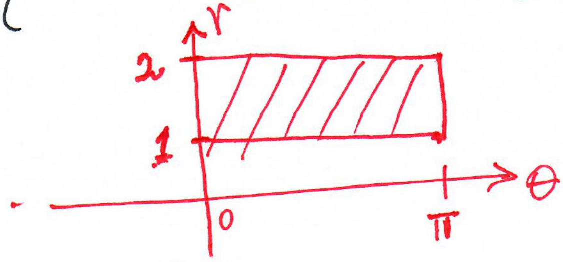
The integral $\int_{-2}^{2} (4-x^2)^{3/2} dx$ not trivial

Need of Polar Coordinates!

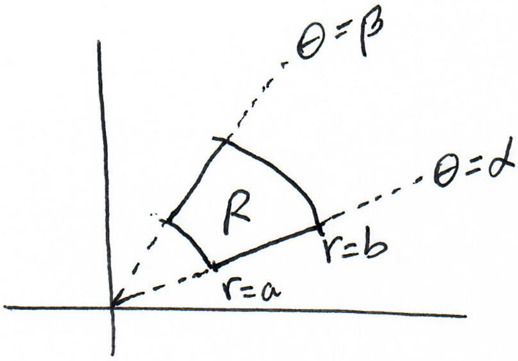
The domain R is a particular case of a "polar rectangle".



$$R = \{ 1 \le r \le 2, \ 0 \le \theta \le \pi \}$$

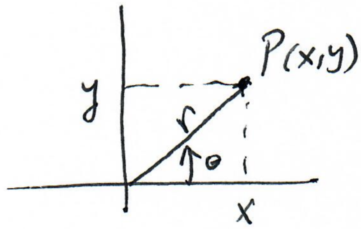


In general, a polar rectangle R is a region as



$$R = \{(r,\theta) \in \mathbb{R}^2 : a \le r \le b, \ \alpha \le \theta \le \beta\}$$

Coordinate transformation from Rectangular to Polar coordinates.



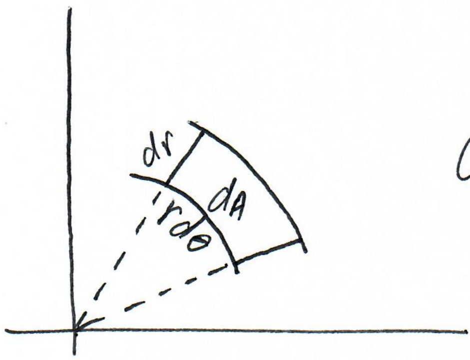
$$(r,\theta) \xrightarrow{T} (x,y)$$

$x = r\cos\theta$
$y = r\sin\theta$

<u>Thm.</u> If $f(x,y)$ is continuous on a polar rectangle R

Then $$\iint_R f(x,y)\,dA = \int_{\theta=\alpha}^{\beta} \int_{r=a}^{b} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

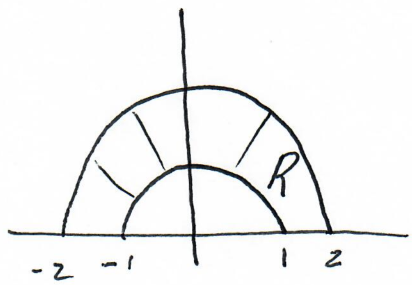
In the previous thm dA can be thought as an infinitesimal polar rectangle with dimensions "$r\,d\theta$" and "dr." In fact,



$$dA \simeq (r\,d\theta)\,dr$$

Back to our example 1

$$I = \iint_R (3x+4y^2)\,dA$$



Using Polar coords.

$$\int_{\theta=0}^{\pi}\int_{r=1}^{2} (3r\cos\theta + 4r^2\sin^2\theta)\,r\,dr\,d\theta$$

$x = r\cos\theta$
$y = r\sin\theta$
$dA = r\,dr\,d\theta$

$$= \int_0^{\pi} \left(r^3\cos\theta + r^4\sin^2\theta\right)\Big|_1^2 d\theta = \int_0^{\pi} \left[8\cos\theta + 16\sin^2\theta - (\cos\theta + \sin^2\theta)\right] d\theta$$

$$= \int_0^{\pi} \left[7\cos\theta + 15\sin^2\theta\right] d\theta = 7\sin\theta\Big|_0^{\pi} + 15\int_0^{\pi}\sin^2\theta\,d\theta$$

$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \sin^2\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

$\Rightarrow \sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

Now, $\displaystyle\int_0^{\pi} \sin^2\theta \, d\theta =$

$$= \int_0^{\pi} \left[\frac{1}{2} - \frac{\cos 2\theta}{2}\right] d\theta =$$

$$= \frac{1}{2}\pi - \frac{1}{2} \left.\frac{\sin 2\theta}{2}\right|_0^{\pi} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

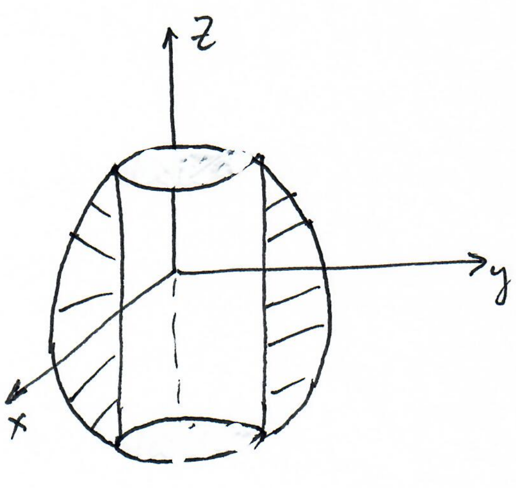
Therefore,

$$I = 7\sin\theta\Big|_0^{\pi} + 15\left(\frac{\pi}{2}\right) = 0 + \frac{15\pi}{2} = \frac{15\pi}{2}$$

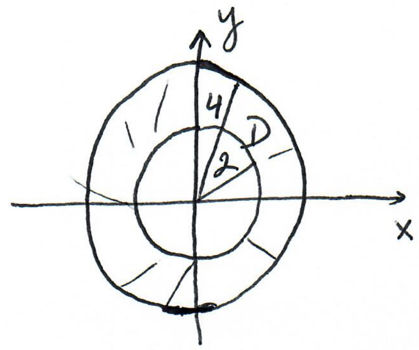
15.4 # 22) Use polar coordinates to find the Volume of the given Solid:

inside the sphere: $x^2+y^2+z^2=16$

and outside the Cylinder: $x^2+y^2=4$



Domain of integration:



$$D = \{ (r,\theta) : 2 \le r \le 4, \quad 0 \le \theta \le 2\pi \}$$

Half-Sphere: $z = +\sqrt{16-(x^2+y^2)}$ using polar coords. $x = r\cos\theta$, $y = r\sin\theta$

$$= \sqrt{16-r^2}$$

and

$$\text{Vol} = 2\iint_D \sqrt{16-(x^2+y^2)}\, dA$$

Polar coords

$$= 2\int_{\theta=0}^{2\pi}\int_{r=2}^{4} \sqrt{16-r^2}\, r\,dr\,d\theta \quad \overset{\substack{u = 16-r^2 \\ du = -2r\,dr}}{=\!=}$$

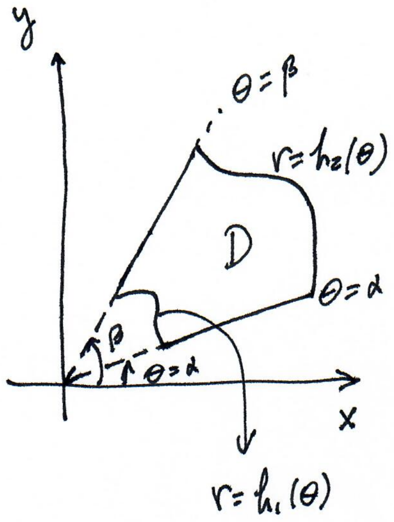
$$= \frac{-2}{2}\int_0^{2\pi}\left(\int_{12}^{0} u^{1/2}\,du\right)d\theta = -\int_0^{2\pi} \frac{2}{3} u^{3/2}\Big|_{12}^{0}\, d\theta \qquad 12^{3/2} = 12\sqrt{12}$$

$$= -\frac{2}{3}\int_0^{2\pi}\left(-12\sqrt{12}\right)d\theta = 8\sqrt{12}\,(2\pi) = 16\sqrt{12}\,\pi$$

$$= \underline{32\sqrt{3}\,\pi}$$

15.4 (Cont)

General Polar Regions.



Thm.- f conts on a polar region

$$D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

$$\iint_D f(x,y)\, dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

θ r

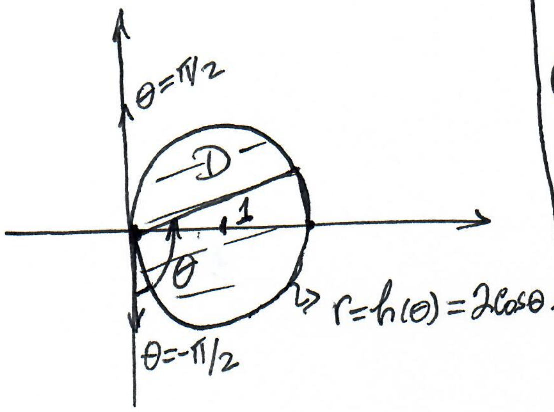
Example 4 (book) Show graph from Stewart's disk.

Vol. of solid under paraboloid : $z = x^2 + y^2$ above xy-plane and inside cylinder : $x^2 + y^2 = 2x$.

Region of integration: D = Project. of cylinder into xy-plane

$$x^2 + y^2 = 2x \iff (x-1)^2 + y^2 = 1$$

Region of Integration



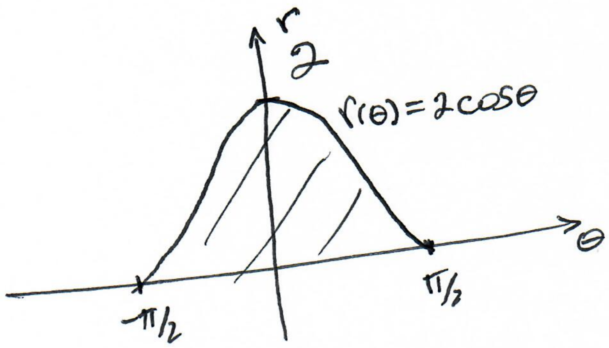
$x^2+y^2=2x$
$\downarrow$
$(x-1)^2+y^2=1$

by introducing polar coordinates:
$x=r\cos\theta$, $y=r\sin\theta$

$$x^2+y^2=2x \Rightarrow r^2=2r\cos\theta, \quad 0\le\theta\le 2\pi$$

or $\boxed{r(\theta)=2\cos\theta,}$ $\quad -\pi/2\le\theta\le\pi/2$

Region of Integration in polar coords.



Thus,

$$I=\iint_D (x^2+y^2)\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\, r\,dr\,d\theta$$

$\theta \quad r$

Then, $\iint\limits_D (x^2+y^2)\,dA = \int\limits_{\underset{\theta}{-\pi/2}}^{\pi/2}\int\limits_{\underset{r}{0}}^{2\cos\theta} r^2\, r\,dr\,d\theta =$

$$= [\cos^2\theta]^2$$

$$= \int_{-\pi/2}^{\pi/2} \frac{r^4}{4}\Big|_0^{2\cos\theta} d\theta = \int_{-\pi/2}^{\pi/2}\frac{16}{4}\cos^4\theta\, d\theta = 4\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \int_{-\pi/2}^{\pi/2}(1+2\cos 2\theta+\cos^2 2\theta)\,d\theta = \pi + \sin 2\theta\Big|_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2}\cos^2 2\theta\, d\theta$$

$$= \pi + 0 + \int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 4\theta}{2}\right)d\theta = \pi + \frac{\pi}{2} + 0 = \frac{3\pi}{2}$$

$$\int_{-\pi/2}^{\pi/2}\cos 4\theta = \frac{\sin 4\theta}{4}\Big|_{-\pi/2}^{\pi/2} = 0$$

Rmk: $\cos^2+\sin^2=1$ and $\cos 2\theta = \cos^2\theta-\sin^2\theta \Rightarrow \cos 2\theta = 2\cos^2\theta - 1$

$\Rightarrow \cos^2\theta = \frac{1+\cos 2\theta}{2}$

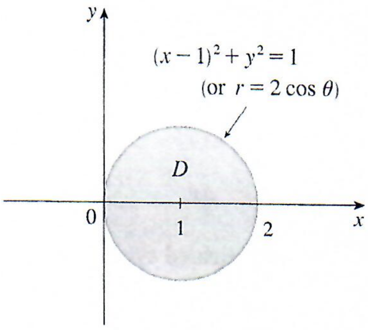


FIGURE 9

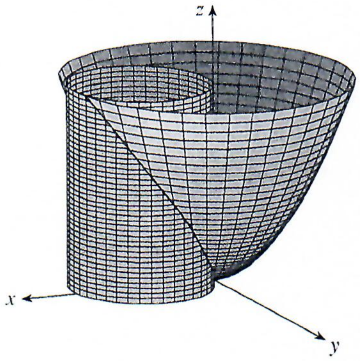


FIGURE 10

EXAMPLE 4 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

SOLUTION The solid lies above the disk D whose boundary circle has equation $x^2 + y^2 = 2x$ or, after completing the square,

$$(x-1)^2 + y^2 = 1$$

(See Figures 9 and 10.)

In polar coordinates we have $x^2 + y^2 = r^2$ and $x = r\cos\theta$, so the boundary circle becomes $r^2 = 2r\cos\theta$, or $r = 2\cos\theta$. Thus the disk D is given by

$$D = \{(r, \theta) \mid -\pi/2 \leqslant \theta \leqslant \pi/2,\ 0 \leqslant r \leqslant 2\cos\theta\}$$

and, by Formula 3, we have

$$V = \iint_D (x^2 + y^2)\, dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2\, r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta$$

$$= 4\int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta = 8\int_0^{\pi/2} \cos^4\theta\, d\theta = 8\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

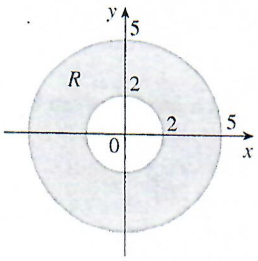
$$= 2\int_0^{\pi/2} \left[1 + 2\cos 2\theta + \tfrac{1}{2}(1 + \cos 4\theta)\right] d\theta$$

$$= 2\left[\tfrac{3}{2}\theta + \sin 2\theta + \tfrac{1}{8}\sin 4\theta\right]_0^{\pi/2} = 2\left(\frac{3}{2}\right)\left(\frac{\pi}{2}\right) = \frac{3\pi}{2}$$

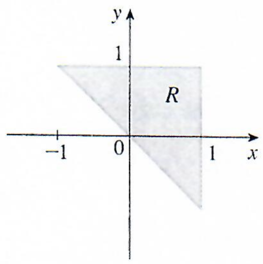
15.3 EXERCISES

1–4 A region R is shown. Decide whether to use polar coordinates or rectangular coordinates and write $\iint_R f(x, y)\, dA$ as an iterated integral, where f is an arbitrary continuous function on R.

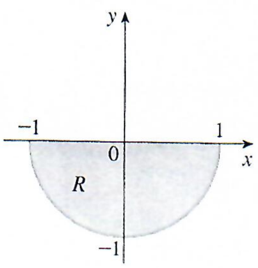
1.



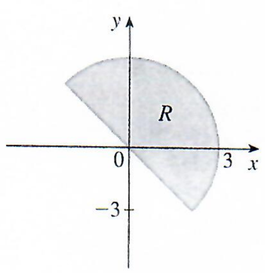
2.



3.



4.



5–6 Sketch the region whose area is given by the integral and evaluate the integral.

5. $\int_{\pi/4}^{3\pi/4} \int_1^2 r\, dr\, d\theta$

6. $\int_{\pi/2}^{\pi} \int_0^{2\sin\theta} r\, dr\, d\theta$

7–14 Evaluate the given integral by changing to polar coordinate

7. $\iint_D x^2 y\, dA$, where D is the top half of the disk with center th origin and radius 5

8. $\iint_R (2x - y)\, dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$

9. $\iint_R \sin(x^2 + y^2)\, dA$, where R is the region in the first quadr between the circles with center the origin and radii 1 and 3

10. $\displaystyle\iint_R \frac{y^2}{x^2 + y^2}\, dA$, where R is the region that lies between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ with $0 < a < b$

11. $\iint_D e^{-x^2-y^2}\, dA$, where D is the region bounded by the semicircle $x = \sqrt{4 - y^2}$ and the y-axis