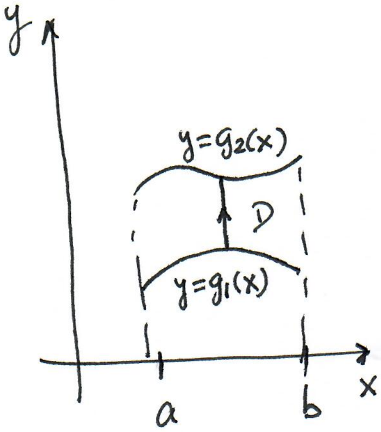
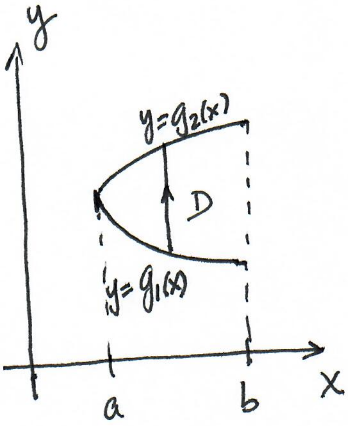
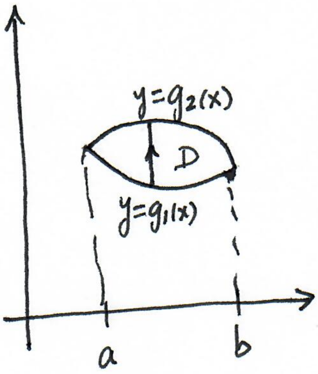
15.2 Double Integrals over General Regions.

Type I regions:





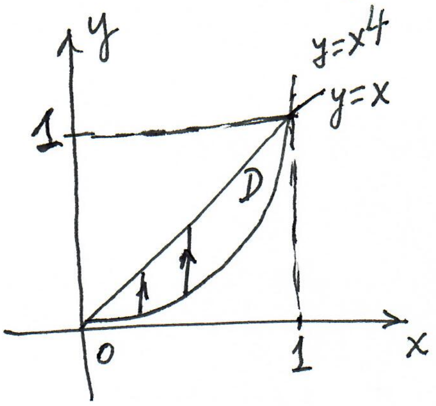


$$D = \left\{ (x,y) \mid a \leq x \leq b, \quad g_1(x) \leq y \leq g_2(x) \right\}$$

Theorem. If f is continuous on a type I region D then,

$$\iint_D f(x,y)\,dA = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx$$

Example1 (Type I region).



$$D=\{(x,y)\mid 0\le x\le 1,\quad x^4\le y\le x\}$$

$$\iint\limits_D (x+2y)\,dA = \int_{\underset{x}{0}}^{1}\int_{\underset{y}{x^4}}^{x}(x+2y)\,dy\,dx$$

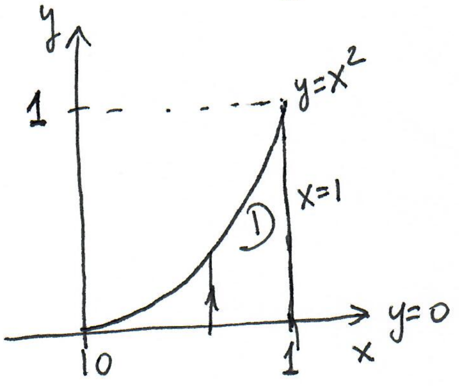
$$= \int_{\underset{x}{0}}^{1}(xy+y^2)\Big|_{y=x^4}^{y=x} dx = \int_{\underset{x}{0}}^{1}[x^2+x^2-x^5-x^8]\,dx$$

$$= \cdots = \frac{7}{18}.$$

* * *

Example 2 (15.2 #17)

Find $\iint\limits_D x\cos y\,dA$, where $D=\left\{\begin{array}{l}\text{region bounded by}\\ y=0,\ y=x^2\\ \text{and}\ \ x=1\end{array}\right\}$.

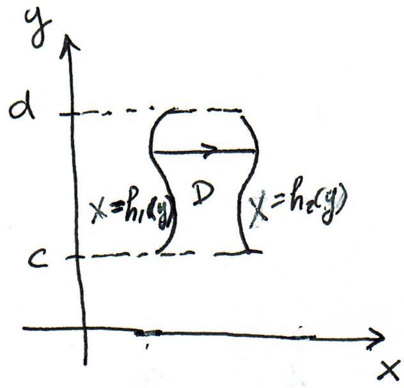


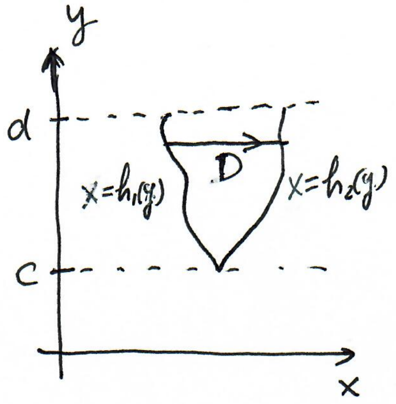
$$\iint\limits_D x\cos y\,dA = \int_{\underset{x}{0}}^{1}\int_{\underset{y}{0}}^{x^2} x\cos y\,dy\,dx$$

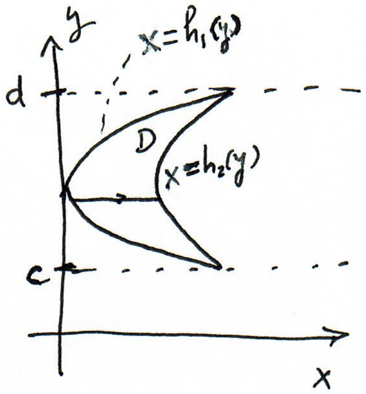
$$= \int_0^1 x\sin y\Big|_{y=0}^{y=x^2} dx = \int_0^1 x\sin x^2\,dx$$

$$= \frac{1}{2}\int_0^1 2x\sin x^2\,dx = \frac{-1}{2}\cos x^2\Big|_0^1 = \frac{-1}{2}(\cos(1)-1).$$

Type II regions:







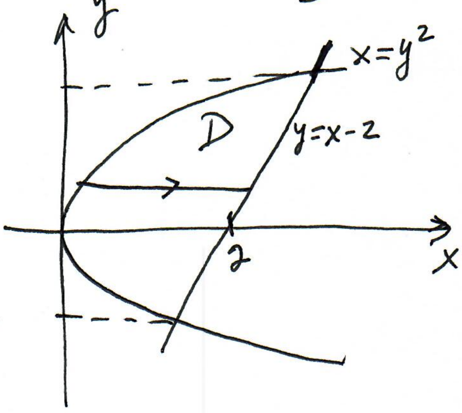
$$D = \{ (x,y) \mid \; c \le y \le d, \quad h_1(y) \le x \le h_2(y) \}$$

theorem. If f is continuous on a type II region D then,

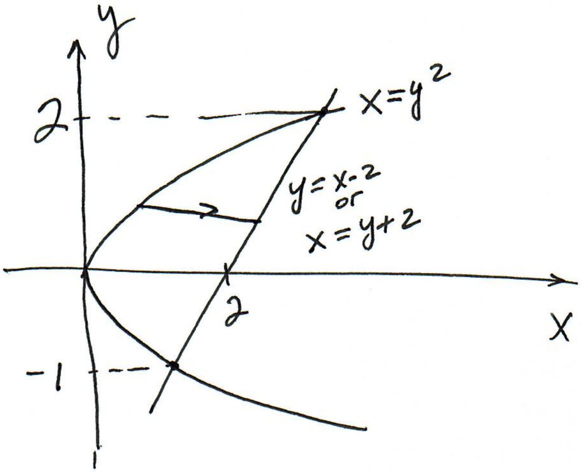
$$\iint_D f(x,y)\,dA = \underset{y}{\int_c^d} \underset{x}{\int_{h_1(y)}^{h_2(y)}} f(x,y)\,dx\,dy$$

Example 3. (15.2 #15)

Evaluate $\iint_D y\,dA$, where D is bounded by $y=x-2$ and $x=y^2$.



Why is this a type II region and why not a type I region?



Intersection

$$y^2=y+2$$
$$y^2-y-2=(y-2)(y+1)=0$$
$$y=2, \quad y=-1$$

$$I=\iint\limits_D y\,dA=\underset{y}{\int_{-1}^{2}}\underset{x}{\int_{y^2}^{y+2}} y\,dx\,dy=\int_{-1}^{2} xy\Big|_{x=y^2}^{x=y+2} dy$$

$$=\int_{-1}^{2}\left[(y+2)y-y^2y\right]dy=\underset{y}{\int_{-1}^{2}}\left(y^2+2y-y^3\right)dy$$

$$=\frac{y^3}{3}+y^2-\frac{y^4}{4}\Big|_{-1}^{2}=\frac{8}{3}+\cancel{4}-\cancel{4}+\frac{1}{3}-1+\frac{1}{4}=\frac{9}{3}-\frac{3}{4}$$

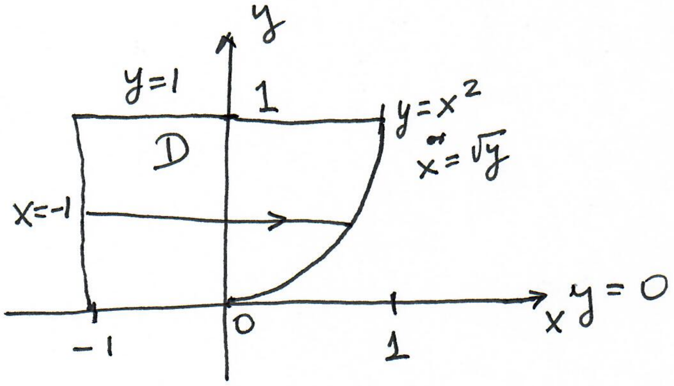
$$=\frac{36-9}{12}=\frac{27}{12}=\frac{9}{4}.$$

Example 4 (Type II region)

Why Type II and not type I. ?

Evaluate $I = \iint\limits_{D} xy \, dA$

Where D region bounded by
$y = x^2$
$y = 0$
$y = 1$
$x = -1$

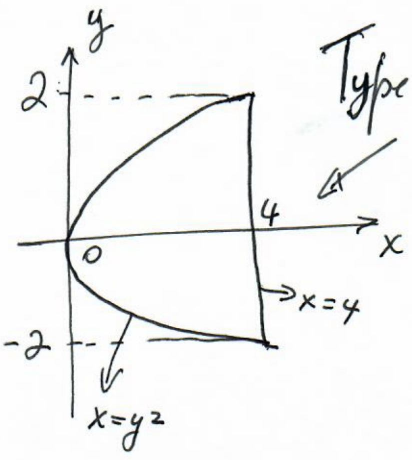


$$I = \underset{y}{\int_0^1} \underset{x}{\int_{-1}^{\sqrt{y}}} xy \, dx \, dy = \int_0^1 y \frac{x^2}{2} \Big|_{x=-1}^{x=\sqrt{y}} dy =$$

$$= \int_0^1 y\left(\frac{y}{2} - \frac{1}{2}\right) dy = \frac{1}{2}\int_0^1 (y^2 - y) \, dy$$

$$= \frac{1}{2}\left(\frac{y^3}{3} - \frac{y^2}{2}\right)\Big|_0^1 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2}\right) = \frac{1}{2}\left(-\frac{1}{6}\right) = -\frac{1}{12}$$

Example 1.- $\iint_R y e^{x^2} dA$



Type I or type II

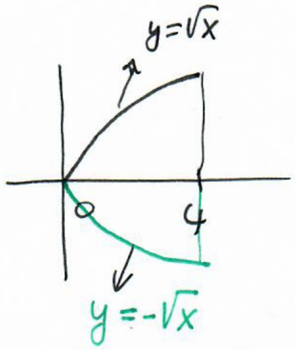
As a Type II

$$= \int_{y=-2}^{y=2} \int_{x=y^2}^{x=4} y e^{x^2} dx\, dy$$

$$= \int_{-2}^{2} \int_{x=y^2}^{x=4} y e^{x^2} dx\, dy$$

No antiderivative for $\int y e^{x^2} dx$!

Then, change order of integration for improvement



As a Type I

$$\int_{x=0}^{4} \int_{y=-\sqrt{x}}^{\sqrt{x}} y e^{x^2} dy\, dx =$$

$$= \int_0^4 \frac{y^2}{2} e^{x^2} \Big|_{-\sqrt{x}}^{\sqrt{x}} dx =$$

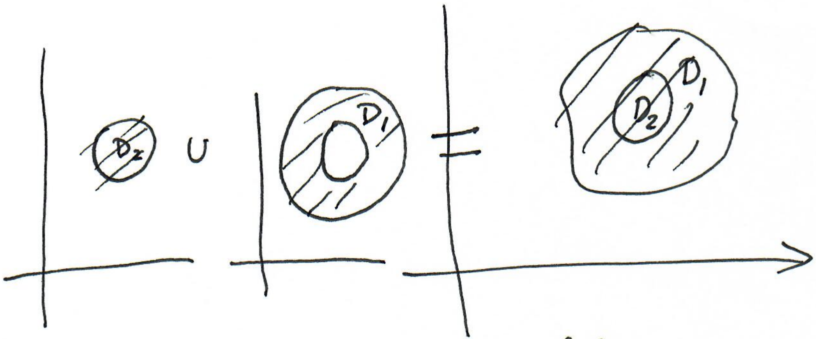
$$= \int_0^4 \left(\frac{x}{2} - \frac{x}{2}\right) e^{x^2} dx = 0.$$

It makes sense!
See symmetry involved.

An important property of Double integrals.

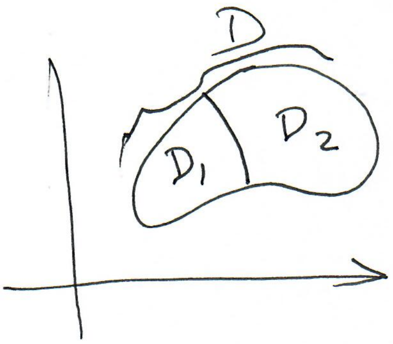
Consider a domain

$$D = D_1 \cup D_2$$

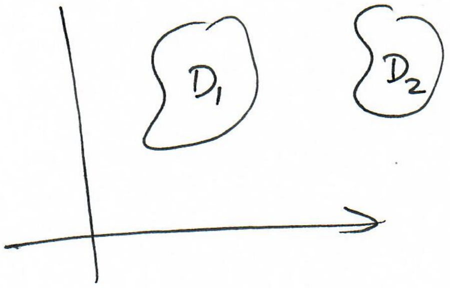


D_1 and D_2 do not overlap except at their boundaries.

$$\iint_D f(x,y)\,dA = \iint_{D_1 \cup D_2} f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA.$$



or



Then, $$\iint_{D_1} f(x,y)\,dA = \iint_D f(x,y)\,dA - \iint_{D_2} f(x,y)\,dA$$