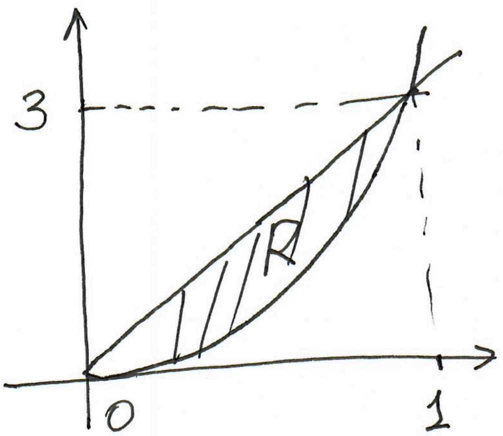
Double integrals used to compute areas of regions between two curves.

Example: $y = f(x) = 3x^2$

$y = g(x) = 3x$



$3x^2 = 3x \Rightarrow \begin{matrix} x = 0 \\ x = 1 \end{matrix}$

Ⓘ Area of R as a 1-D integral $= \int_0^1 (3x - 3x^2)dx = \left(\frac{3}{2}x^2 - x^3\right)\Big|_0^1$

$= \frac{3}{2} - 1 = \frac{1}{2}.$

Equivalent to:

Ⓘⓘ Type I integration in $\mathbb{R}^2$:

$$\underset{x}{\int_0^1} \left(\int_{\underset{y}{3x^2}}^{3x} dy \right) dx = \int_0^1 y \Big|_{y=3x^2}^{y=3x} dx = \int_0^1 [3x - 3x^2] dx$$

Ⅲ It can also be seen as

<u>Type II integration in $\mathbb{R}^2$:</u>

$$\int_{\substack{0\\y}}^{3} \left(\int_{x=\frac{1}{3}y}^{x=(\frac{1}{3}y)^{1/2}} dx \right) dy$$

$y = 3x^2$
$\Rightarrow x = \left(\frac{1}{3}y\right)^{1/2}$

$y = 3x$
$\Rightarrow x = \frac{1}{3}y$

$$= \int_0^3 \left[\left(\frac{1}{3}y\right)^{1/2} - \frac{1}{3}y \right] dy$$

$$= \left[\frac{2}{\cancel{3}} \left(\frac{1}{3}y\right)^{3/2} \cancel{3} - \frac{1}{3} \frac{y^2}{2} \right] \Bigg|_0^3$$

$$= 2\left(\frac{1}{\cancel{3}}\cancel{3}\right)^{3/2} - \frac{1}{\cancel{3}} \frac{3^{\cancel{2}}}{2} = 2 - \frac{3}{2} = \frac{1}{2}$$

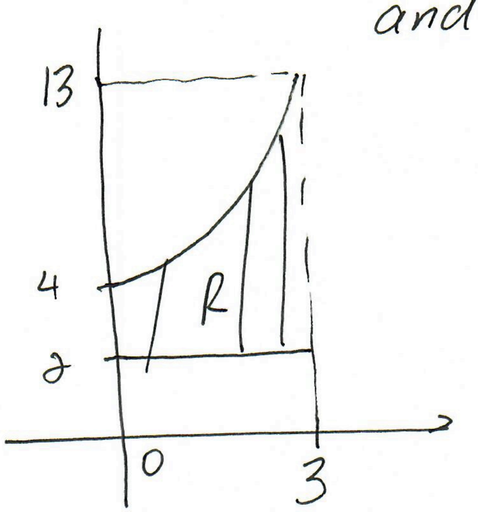
Compute area of region R

R is region enclosed by $y = x^2 + 4$

and $y = 2$

$x = 0$

$x = 3$



As a double integral:

$$\iint_R dA = \int_0^3 \int_{y=2}^{y=x^2+4} dy\,dx$$

$$= \int_0^3 y\Big|_2^{x^2+4} dx = \int_0^3 (x^2+4\underset{2}{\underline{-2}})\,dx$$

$$= \frac{x^3}{3} + 2x\Big|_0^3 = 9 + 6 = \underline{\underline{15}}$$

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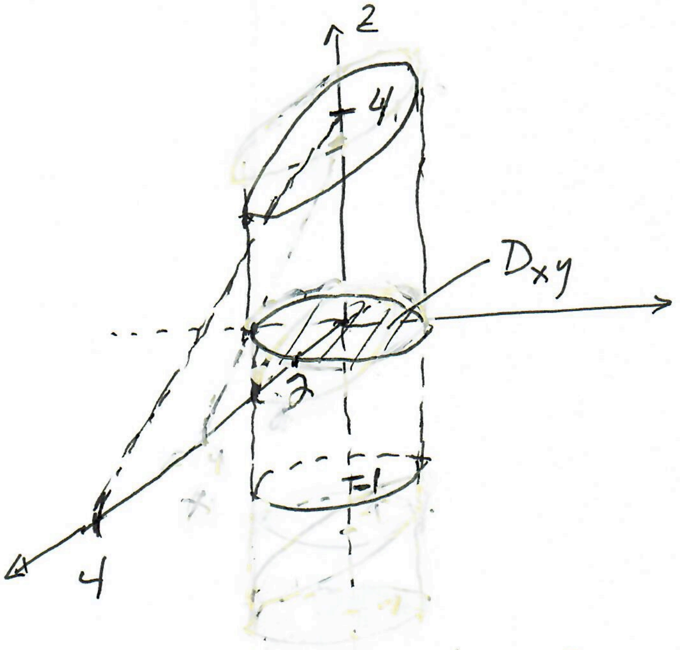
Triple Integrals to compute Volume between two Surfaces.

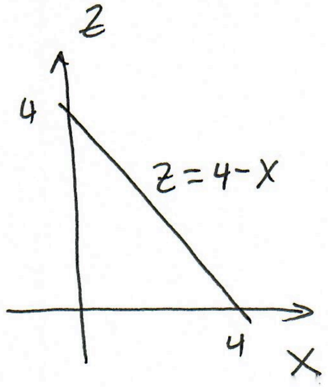
Example: Compute Volume enclosed by

Cylinder: $x^2+y^2=4$,

plane 1: $z=-1$

plane 2: $z+x=4 \Rightarrow z=4-x$.





As a double integral:

$$\iint\limits_{\substack{D_{xy}\\ \text{Circle}}} (4-x-(-1))\,dA = \int_0^{2\pi}\int_0^2 (4-x+1)\,r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^2 (5 - r\cos\theta)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\int_0^2 (5r - r^2\cos\theta)\,dr\right)d\theta$$

$$= \int_0^{2\pi}\left(\frac{5r^2}{2} - \frac{r^3}{3}\cos\theta\right)\Big|_0^2 d\theta$$

$$= \int_0^{2\pi}\left(\frac{5(4)}{2} - \frac{8}{3}\cos\theta\right)d\theta$$

$$= \int_0^{2\pi}\left(10 - \frac{8}{3}\cos\theta\right)d\theta$$

$$= \left(10\theta + \frac{8}{3}\sin\theta\right)\Big|_0^{2\pi} = 20\pi.$$

It can be written as a triple integral

$$\underset{\theta}{\int_0^{2\pi}}\ \underset{r}{\int_0^2}\ \underset{z}{\int_{-1}^{4-x}} dz\, r\,dr\,d\theta = \underset{\theta}{\int_0^{2\pi}}\ \underset{r}{\int_0^2} z\Big|_{-1}^{4-x} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 [4 - x - (-1)]\, r\,dr\,d\theta. \quad ✓$$

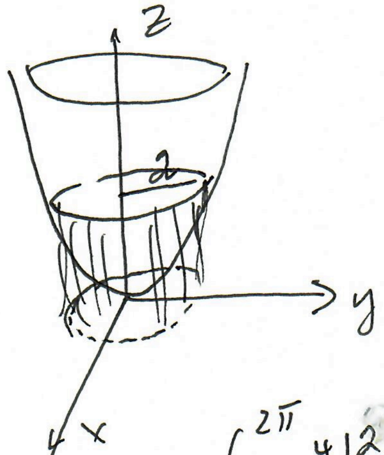
Before Example 4.

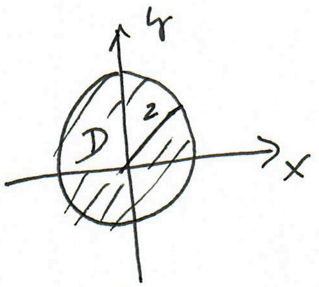
Find Vol. enclosed ~~by~~ below the paraboloid

$$z = x^2+y^2$$

and inside the cylinder $x^2+y^2=4$ above xy-plane.

— ○ —





Ⓘ $$\int_0^{2\pi}\int_0^{2}\big(\overset{=r^2}{x^2+y^2}\big)\, r\,dr\,d\theta = \int_0^{2\pi} \frac{r^4}{4}\Big|_0^2 d\theta = \int_0^{2\pi} 4\,d\theta = \underline{\underline{8\pi}}.$$

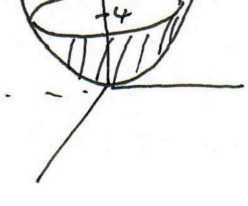
θ r

As a Volume integral $= \int_0^{2\pi}\int_0^{2}\left(\int_{z=0}^{z=x^2+y^2} dz\right) r\,dr\,d\theta$

Same as Vol. enclosed under the paraboloid $z = x^2+y^2$ and below $z=4$ plane and above the xy-plane.

Different than Vol. enclosed by the paraboloid and below the plane $z=4$

Ⓘⓘ $$\int_0^{2\pi}\int_0^{2}\int_{z=x^2+y^2}^{z=4} dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^{2}\big[4-(\overset{=r^2}{x^2+y^2})\big]\, r\,dr\,d\theta$$



Compute Volume of region R
enclosed by paraboloid:

$z = x^2+y^2+4$

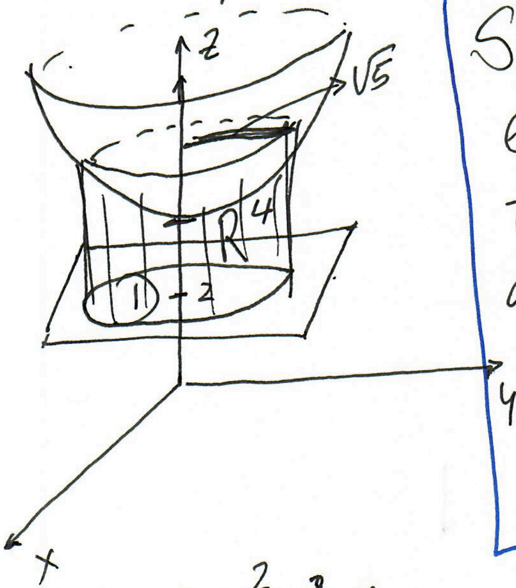
and Cylinder: $x^2+y^2=5$

and above the plane $z=2$

ask for this one

Same as Volume enclosed under the paraboloid above the plane $z=2$ and below the plane $z=9$

Combining
$z = x^2+y^2+4$
with
$z = 9$
$9 = x^2+y^2+4$
or $\boxed{x^2+y^2=5}$



$$\iiint_R dv = \iint_D \int_{\substack{2\\z}}^{z=x^2+y^2+4} dz\,dA$$

using Cylindrical coords.

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$= \int_{\substack{0\\\theta}}^{2\pi}\int_{\substack{0\\r}}^{\sqrt5}\int_{z=2}^{z=r^2+4} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt5} z\Big|_2^{r^2+4} r\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt5}\left(r^2+\overbrace{4-2}^{2}\right) r\,dr\,d\theta$$

8A

$$= \int_0^{2\pi} \frac{r^4}{4} + r^2 \Big|_0^{\sqrt{5}} d\theta = \int_0^{2\pi}$$

27
3
81
36
117

$$= \int_0^{2\pi} \left(\frac{5^2}{4} + 5\right) d\theta =$$

$$= \int_0^{2\pi} \left(\frac{25}{4} + \frac{20}{4}\right) d\theta = \boxed{\frac{45}{4} 2\pi} = \boxed{\frac{45}{2}\pi}$$