13.2 Vector Functions

Exercise

$$\lim_{t\to 0} \vec{r}(t) = \lim_{t\to 0} \left\langle 1+t^3, t e^{-t}, \frac{\sin t}{t} \right\rangle$$

$$= \left\langle \lim_{t\to 0} (1+t^3), \lim_{t\to 0} t e^{-t}, \lim_{t\to 0} \frac{\sin t}{t} \right\rangle$$

$$= \langle 1, 0, 1 \rangle.$$ Is $\vec{r}(t)$ Continuos at $t=0$?

Derivative

For real valued functions : $f: D \subset \mathbb{R} \to \mathbb{R}$

For $x_0 \in D$, $$f'(x_0) = \frac{df}{dx}(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

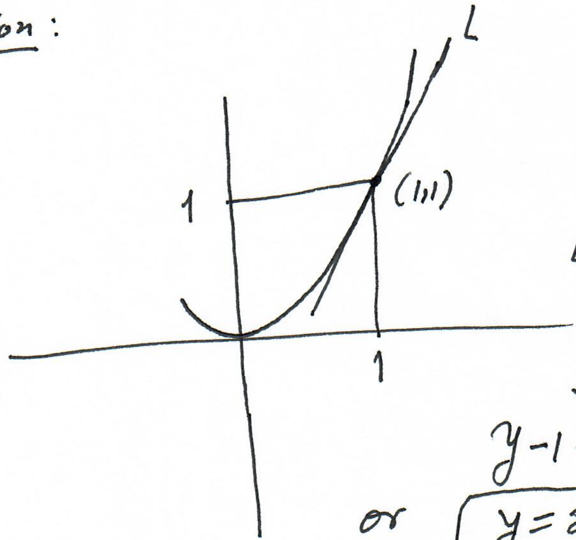
Geometric Interpretation:

Consider

$f(x) = x^2$

then, $f'(x) = 2x$

at $x=1$ $f'(1) = 2$



Tangent line to the graph of f at $(1,1)$.

$L: y-1 = f'(1)(x-1)$

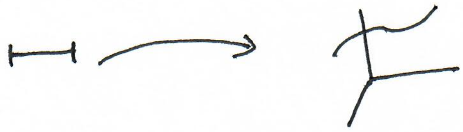
$y-1 = 2(x-1)$

or $\boxed{y = 2x-1}$

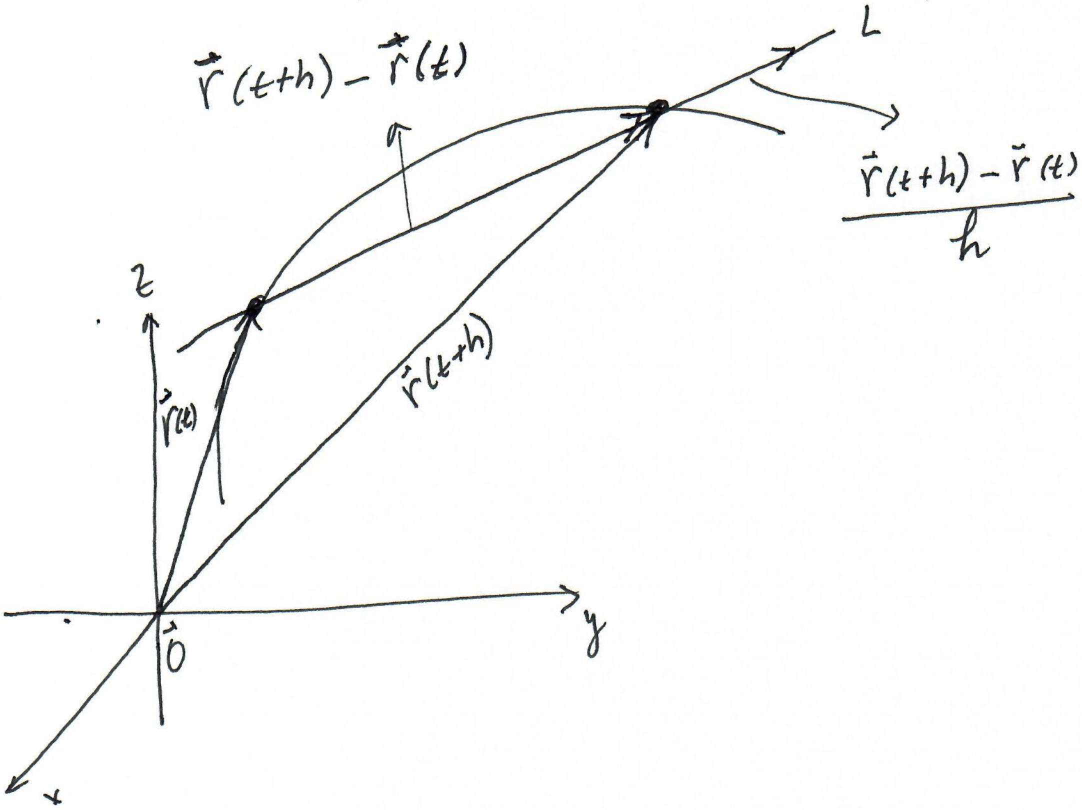
Tangent Vector to space Curve at a point $P_0\ (x_0, y_0, z_0)$.

Consider $\vec{r}: I \subset \mathbb{R} \to \mathbb{R}^3$

$$t \longrightarrow \vec{r}(t)$$



If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ has this graph



Def.- If $\lim\limits_{h \to 0} \dfrac{\vec{r}(t+h) - \vec{r}(t)}{h}$ exists, we <u>denote it as $\vec{r}\,'(t)$</u>

and we call it, the derivative of the vector function $\vec{r}(t)$ at t.

Go to chapter 13 document. ⟶

Tangent vector to space curve $\mathcal{C}$: $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$a \le t \le b$

Tangent line of $\mathcal{C}$ at $P_0(x_0, y_0, z_0)$.

Tangent vector: $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$

at any $t \in (a, b)$

Tangent line of $\mathcal{C}$ at $P_0(x_0, y_0, z_0)$:

If $t = t_0$ is such that

$$\vec{r}(t_0) = \langle x(t_0), y(t_0), z(t_0) \rangle = \langle x_0, y_0, z_0 \rangle$$

then,

Tangent line L: $\langle x, y, z \rangle = t \langle x'(t_0), y'(t_0), z'(t_0) \rangle + \langle x_0, y_0, z_0 \rangle$.

assuming $x'(t_0)$, $y'(t_0)$, and $z'(t_0)$ exist.

13.2 # 24. Tangent line

$\vec{r}(t) = \langle e^t, te^t, te^{t^2} \rangle$ at $(1,0,0)$.

$\vec{r}'(t) = \langle e^t, e^t + te^t, e^{t^2} + 2t^2 e^{t^2} \rangle$ $\quad \vec{r}(0) = \langle 1,0,0 \rangle$

$r'(0) = \langle 1,1,1 \rangle$

Line L is $\boxed{\langle x,y,z \rangle = t \langle 1,1,1 \rangle + \langle 1,0,0 \rangle}$

or $x = t+1, \quad y = t, \quad z = t.$

If $|\vec{r}(t)| = c$

$\Rightarrow \vec{r}'(t) \perp \vec{r}(t).$

Proof.

$$\vec{r}(t) \cdot \vec{r}(t) = |\vec{r}(t)|^2 = c^2 \qquad (1)$$

Now,

$$\frac{d}{dt}\left(\vec{r}(t) \cdot \vec{r}(t)\right) = \vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t) = 2\,\vec{r}'(t) \cdot \vec{r}(t).$$

$$\Rightarrow \frac{d}{dt}(1) \Rightarrow 0 = 2\,\vec{r}'(t) \cdot \vec{r}(t) \Rightarrow \boxed{\vec{r}'(t) \cdot \vec{r}(t) = 0.}$$

Go to Chap 13 (LaTex) document

13.2 # 46

Prove:

$$\frac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\, u'(f(t)).$$

Proof: Let $\vec{u}(t) = \langle u_1(t), u_2(t), u_3(t) \rangle$

Then $\vec{u}(f(t)) = \langle u_1(f(t)), u_2(f(t)), u_3(f(t)) \rangle$

and

$$\frac{d}{dt}\left[\vec{u}(f(t))\right] = \left\langle \frac{d}{dt}\left[u_1(f(t))\right], \frac{d}{dt}\left[u_2(f(t))\right], \frac{d}{dt}\left[u_3(f(t))\right] \right\rangle$$

Using chain rule for real valued fn.

$$= \langle u_1'(f(t))\, f'(t), u_2'(f(t))\, f'(t), u_3'(f(t))\, f'(t) \rangle$$

$$= f'(t) \langle u_1'(f(t)), u_2'(f(t)), u_3'(f(t)) \rangle.$$

$$= f'(t)\, \vec{u}'(f(t)).$$

Exercise Given $\vec{u}(t) = \left\langle e^t, t^2, t^3 \right\rangle$, $t \in \mathbb{R}$

and $f(t) = \sin t$

Find $\frac{d}{dt}\left[\vec{u}(f(t))\right]$.

Answ:

$$\frac{d\vec{u}}{dt}(t) = \vec{u}'(t) = \left\langle e^t, 2t, 3t^2 \right\rangle$$

$$\frac{df}{dt}(t) = f'(t) = \cos t.$$

Then applying the chain rule: $\left[u(f(t))\right]' = f'(t)\,\vec{u}'(f(t))$

$$\frac{d}{dt}\left[\vec{u}(f(t))\right] = \cos t \left\langle e^{\sin t}, 2\sin t, 3\sin^2 t \right\rangle.$$

Integrals

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt,\ \int_a^b h(t)\,dt \right\rangle$$

remind some important integration formulas:

\# 34) $\int_0^1 \left(\frac{4}{1+t^2}\hat{j} + \frac{2t}{1+t^2}\hat{k}\right)dt$

$= 4 \arctan t \Big|_0^1 \hat{j} + \ln(1+t^2)\Big|_0^1 \hat{k} =$

$= 4\left[\arctan 1 - \arctan 0\right]\hat{j} + \left[\ln(1+1) - \ln(1+0)\right]\Big|_0^1 \hat{k} =$

$= 4\left(\frac{\pi}{4} - 0\right)\hat{j} + \ln(2)\hat{k} = \langle 0, \pi, \ln(2) \rangle.$

\# Other integrals: $\int t e^t\,dt$

$\int \tan t\,dt$

$\int e^t \sin t\,dt$.