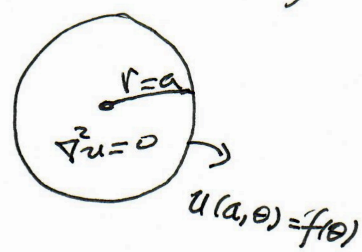
Laplace equation inside Circular Disk.

$$
\begin{cases}
\nabla^2_{r,\theta} u = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0 & (1) \\
\qquad\qquad 0 < r < a, \quad -\pi < \theta < \pi. & (2) \\
u(a,\theta) = f(\theta), \quad -\pi < \theta < \pi. &
\end{cases}
$$



We will solve (1)-(2) using Separation of Variables technique. For this, we need more conditions on u

We need conditions not only at $r=a$

but also $\underline{\underline{r=0}}$, $\underline{\underline{\theta=\pi}}$ and $\underline{\underline{\theta=-\pi}}$

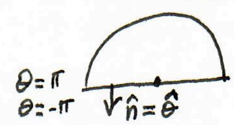
Extra physical conditions are

a) Temperature is finite at $r=0$.

i.e., $|u(0,\theta)| < \infty$.

b) Continuity of temperature and flux at $\underline{\underline{\theta=\pi}}$ Segment

Same as $\theta=-\pi$



$$\boxed{\begin{aligned} u(r,-\pi) &= u(r,\pi). \\ \frac{\partial u}{\partial \theta}(r,-\pi) &= \frac{\partial u}{\partial \theta}(r,\pi) \end{aligned}}$$

Why $\dfrac{\partial u}{\partial \theta}$ and not $\dfrac{\partial u}{\partial r}$.

$\vec{\phi}\cdot\hat{n}(r,\pi) = \vec{\phi}\cdot\hat{n}(r,-\pi)$

$-K_0\,\nabla u\cdot\hat{n}(r,\pi) = -K_0\,\nabla u\cdot\hat{n}(r,-\pi)$

or $\dfrac{\partial u}{\partial n}(r,\pi) = \dfrac{\partial u}{\partial n}(r,-\pi)$

$\hat{n}=\hat{\theta}$ $\quad \dfrac{\partial u}{\partial \theta}(r,\pi) = \dfrac{\partial u}{\partial \theta}(r,-\pi)$

Also, called <u>periodicity</u> conditions.

or interface conditions

$$\left\{\begin{aligned} &\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 && (6.1)\\ &u(a,\theta) = f(\theta). && (6.2)\\ &|u(0,\theta)| < \infty && (6.3)\\ &u(r,-\pi) = u(r,\pi). && (6.4)\\ &\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi) && (6.5) \end{aligned}\right.$$

All B.C's are homogeneous except (6.2).

Therefore, Separation of Variables applies.

$$\boxed{u(r,\theta) = \phi(\theta)\, G(r).}$$

Subst. leads to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\phi(\theta) + \frac{1}{r^2}G(r)\frac{d^2\phi}{d\theta^2} = 0$$

Dividing by $\frac{1}{r^2}G(r)\phi(\theta)$.

$$\frac{r}{G(r)}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi(\theta)}\frac{d^2\phi}{d\theta^2} = +\lambda$$

expect oscillations in θ.

$$\Rightarrow \quad \boxed{\frac{d^2\phi}{d\theta^2} + \lambda\,\phi(\theta) = 0} \qquad (7.1)$$

and

$$\frac{r}{G(r)}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \lambda$$

or

$$\boxed{r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - \lambda\, G(r) = 0.} \qquad (7.2)$$

BC's (6.4) and (6.5) periodicity conds. implies

$$G(r)\,\phi(-\pi) = G(r)\,\phi(\pi) \overset{\text{for non-trivial solns.}}{\Rightarrow} \phi(-\pi)=\phi(\pi).$$

$$G(r)\,\frac{d\phi}{d\theta}(-\pi) = G(r)\,\frac{d\phi}{d\theta}(\pi) \Rightarrow \phi'(-\pi)=\phi'(\pi).$$

Therefore, the eigenvalue problem is given by.

$$\begin{cases} \phi''(\theta) + \lambda\,\phi(\theta) = 0 \\ \phi(-\pi) = \phi(\pi) \\ \phi'(-\pi) = \phi'(\pi). \end{cases}$$

whose eigenvalues are $\lambda_n = \left(\frac{n\pi}{\pi}\right)^2 = n^2, \quad n=0,1,2\ldots$

and corresponding eigenfns. $\hat{\phi}_n(\theta) = \left\{\begin{matrix} 1 \\ \cos(n\theta) \\ \sin(n\theta) \end{matrix}\right\} \quad n=1,2,\ldots$

To obtain the solns. for $G(r)$, we will use the condition: $|u(0,\theta)|<\infty \Rightarrow |\phi(\theta)\,G(0)|<\infty$
Since $|\phi(\theta)|<\infty$ this condition requires $|G(0)|<\infty$.

The solns. for (7.2) satisfy

$$\begin{cases} r^2\dfrac{d^2G_n}{dr^2} + r\dfrac{dG_n}{dr} - \overset{=n^2}{\lambda}\,G_n(r) = 0, \quad n=0,1,2\ldots & (8.1) \\ |G_n(0)| < \infty & (8.2) \end{cases}$$

Case 1) $\lambda = 0 \quad (n=0)$

Then $$\boxed{r^2 \frac{d^2 G_0}{dr^2} + r \frac{dG_0}{dr} = 0}$$

or $$\frac{d^2 G_0}{dr^2} + \frac{1}{r}\frac{dG_0}{dr} = 0 \Leftrightarrow \frac{1}{r}\frac{d}{dr}\left(r \frac{dG}{dr}\right) = 0$$

$$\Rightarrow \frac{d}{dr}\left(r \frac{dG_0}{dr}\right) = 0 \Rightarrow r\frac{dG_0}{dr} = C_1$$

$$\Rightarrow \frac{dG_0}{dr} = \frac{C_1}{r} \Rightarrow \boxed{G_0(r) = C_1 \ln r + C_2}$$

Now, $|G_0(0)| < \infty \overset{C_1=0}{\Rightarrow} \boxed{G_0(r) = C_2}$

Case 2) $\lambda \neq 0, \quad n = 1, 2, \ldots$

$$\boxed{r^2 \frac{d^2 G_n}{dr^2} + r \frac{dG_n}{dr} - n^2 G_n(r) = 0}$$

Euler equation. Look for solns. as $G(r) = r^p$

$$G_n'(r) = p\, r^{p-1}, \quad G_n''(r) = p\,(p-1)\, r^{p-2}$$

Subst. leads to

$$p(p-1)\cancel{r^p} + p\cancel{r^p} - n^2 \cancel{r^p} = 0 \quad \text{or} \quad p(p-1) + p - n^2 = 0$$

$$p^2 - n^2 = 0 \Rightarrow p = \pm n.$$

Thus, $$\boxed{G_n(r) = C_1 r^n + C_2 r^{-n}.}$$

B.C. $\quad |G_n(0)| < \infty \Rightarrow \underline{C_2 = 0}$

As a consequence, $G_n(r) = C_1 r^n$.

Therefore, product solns:

$$A_0 (1), \quad \begin{array}{l} A_n r^n \cos n\theta \\ B_n r^n \sin n\theta \end{array}$$

and infinite series soln. of BVP:

$$u(r,\theta) = \sum_{n=1}^{\infty} \left[A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta)\right] + A_0$$

Using IC.

$$f(\theta) = u(a,\theta) = \sum_{n=1}^{\infty} \left[A_n a^n \cos(n\theta) + B_n a^n \sin(n\theta)\right] + A_0$$

Using orthog.:

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$A_n a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta\, d\theta$$

$$B_n a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta\, d\theta.$$

Before 2.5.4.

Solution for Laplace's equation inside a circular disk.

$$\begin{cases} \nabla^2_{r,\theta} u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 & (1) \\ u(a,\theta) = f(\theta) & (2) \end{cases}$$

$$\nabla^2_{r,\theta} u = \Delta_{r,\theta} u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta)\, r^n + \sum_{n=1}^{\infty} B_n \sin(n\theta)\, r^n \qquad (3)$$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta, \qquad A_n a^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta \qquad (4)$$

$$B_n a^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta \qquad (5)$$

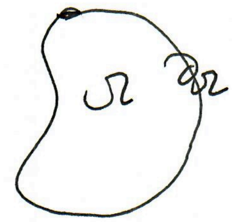
Subst. (4) - (5) into (3).

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta + \sum_{n=1}^{\infty} \frac{1}{\pi}\left(\int_{-\pi}^{\pi} f(\bar\theta)\cos(n\bar\theta)\,d\bar\theta\right)\cos(n\theta)\,\frac{r^n}{a^n}$$

$$+ \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} f(\bar\theta)\sin(n\bar\theta)\,d\bar\theta\right)\sin(n\theta)\,\frac{r^n}{a^n}$$

Qualitative Properties of Laplace Equation

Consider the BVP



$$\begin{cases} \nabla^2 u = 0, & \vec{x} \in \Omega \quad (1) \\ u(\vec{x}_s) = h(\vec{x}_s), & \vec{x}_s \in \partial\Omega \quad (2) \end{cases}$$

where $h(\vec{x}_s)$ is continuous on $\partial\Omega$.

Definition of well-posedness of BVP (1)-(2)

The BVP (1)-(2) is well-posed if

i) It has a unique solution.

ii) This unique solution "depends continuosly on the boundary conditions." It means the solutión doesn't change much for small changes in the boundary conditions.

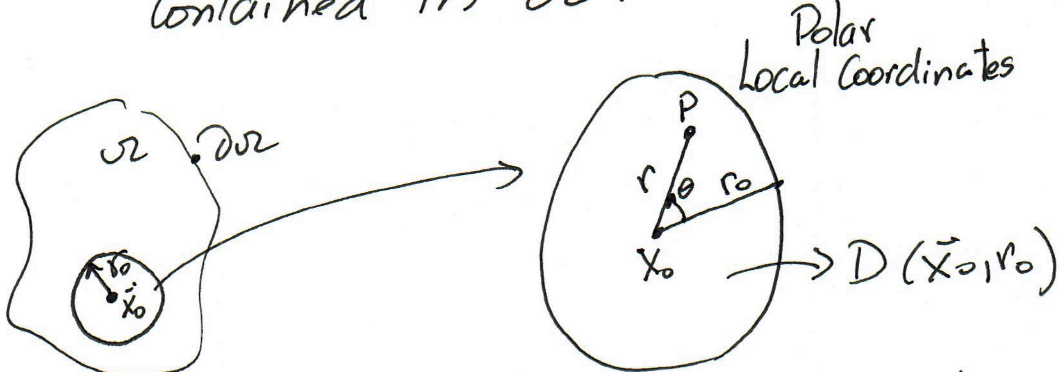
Theorem (Mean Value theorem)

If u is a soln. of (1)-(2), then

$u(\vec{x}_0)$ for $\vec{x}_0 \in \Omega$ (interior points)

is equal to the average of the values of u along any circle of radius r_0 centered at $\vec{x}_0$.

Proof: Consider an arbitrary circle centered at $\vec{x}_0$ of radius r_0, which is contained in Ω.



Next, we define a local polar coord. system with center at $\vec{x}_0$.

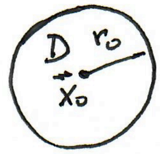
From our previous for the solution of a Dirichlet problem for a Laplace equation in a circular region we know that the Solution can be represented by

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right] r^n,$$

in $D(\vec{x}_0, r_0)$

In particular at $r=0$

$$u(\vec{x}_0) = u(0,\theta) = A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(r_0,\theta)\, d\theta$$



Average of $u(r,\theta)$ along Circle of radius r_0.

<u>Theorem</u> <u>(Max. Pple)</u>

The max and min of the Solution $u(\vec{x})$ of <u>BVP (1)-(2)</u> occur on the boundary $\partial\Omega$.

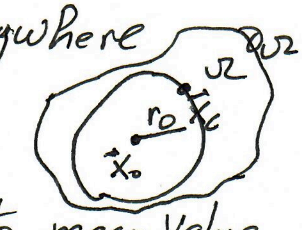
<u>Proof</u>. Next page.

Proof. Two Cases:

i) The Solution u is constant everywhere including the boundaries. In this case the max pple is trivially true.

ii) The Solution is not constant everywhere

Let's consider an arbitrary circle



$B_r(\vec{x}_0) \subset \Omega$. Then according to mean Value theorem

$$u(\vec{x}_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(r_0, \theta)\, d\theta \quad (1.1)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\vec{x}_c)\, d\theta, \quad \vec{x}_c \in \partial\Omega$$

Therefore, if we assume that the max occurs at an internal point $\vec{x}_0 \in \Omega$ (1.1) implies that

i) There is $\vec{x}_c^*$ in $B_r(\vec{x}_0)$ Such that

$$u(x_0) < u(\vec{x}_c^*)$$

Contradicting that $u(\vec{x}_0)$ is max. Value.

or ii) $u(\vec{x}_0) = u(\vec{x}_c)$, for all $\vec{x}_c \in B_r(\vec{x}_0)$

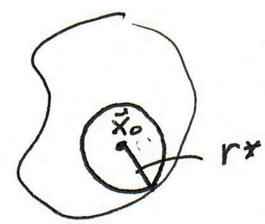
Then, we can try another circle $B_{\bar{r}}(\vec{x}_0) \subset \Omega$
$\bar{r} \neq r$.

if $u(\vec{x}_0) = u(\vec{x}_c^*)$, for all $\vec{x}_c^*$
in $B_{\bar{r}}(\vec{x}_0)$.

and this happens to all circles contained in Ω with center at $\vec{x}_0$,

then, we can construct a circle up to the boundary $\partial\Omega$ (touching it), which implies that the max is reached at the boundary also, because

$$u(\vec{x}_0) = u(\vec{x}^*) \quad \vec{x}^* \in B_{r^*}(\vec{x}_0)$$



Theorem

The BVP (1)-(2) is well-posed.

Proof

Assume $u_1(\vec{x})$ and $u_2(\vec{x})$ are solutions of (1)-(2).

Then, $w(\vec{x}) = u_1(x) - u_2(x)$ Satisfies

$$\boxed{\nabla^2 w = \nabla^2 u_1 - \nabla^2 u_2 = 0 - 0 = 0}$$

and $w(\vec{x}_s) = u_1(\vec{x}_s) - u_2(\vec{x}_s) = h(\vec{x}_s) - h(\vec{x}_s) = 0.$

It means $w(\vec{x})$ satisfies the BVP:

$$\begin{cases} \nabla^2 w = 0, & \vec{x} \in \Omega. \qquad (3) \\ w(\vec{x}_s) = 0, & \vec{x}_s \in \partial\Omega. \qquad (4) \end{cases}$$

Applying the max principle to (3)-(4)

$$\boxed{0 = \min_{\vec{x}_s \in \partial\Omega} (w(\vec{x}_s)) \le w(\vec{x}) \le \max_{\vec{x}_s \in \partial\Omega} (w(\vec{x}_s)) = 0 \quad \text{for all } \vec{x} \in \Omega \cup \partial\Omega}$$

Then, $\boxed{w(\vec{x}) = 0,}$ for all $\vec{x} \in \Omega \cup \partial\Omega$.

As a consequence,

$$\boxed{u_1(\vec{x}) = u_2(\vec{x})}, \quad \vec{x} \in \Omega \cup \partial\Omega.$$

which shows that the solution is unique.

To show the continuous dependence of the boundary condition consider $v(\vec{x})$ satisfying the BVP:

$$\begin{cases} \nabla^2 v = 0, & \vec{x} \in \Omega. & (5) \\ v(\vec{x}_s) = g(\vec{x}_s), & \vec{x}_s \in \partial\Omega & (6) \end{cases}$$

where $g(\vec{x}_s)$ is such that

$$\boxed{|h(\vec{x}_s) - g(\vec{x}_s)| < \varepsilon, \quad \varepsilon > 0} \quad (6.1)$$

then, and

$$\begin{cases} \nabla^2(u - v) = \nabla^2 u - \nabla^2 v = 0 - 0 = 0 \quad (7) & \vec{x} \in \Omega. \\ (u - v)(\vec{x}_s) = u(\vec{x}_s) - v(\vec{x}_s) \quad (8) \\ \quad = h(\vec{x}_s) - g(\vec{x}_s) \text{ on } \partial\Omega. \end{cases}$$

Applying max pple to (7)-(8) and Considering (6.1), i.e, $-\varepsilon < (h(\vec{x}_s) - g(\vec{x}_s)) < \varepsilon$

then,

$$-\varepsilon < \min_{\vec{x}_s \in \partial\Omega} (h(\vec{x}_s) - g(\vec{x}_s)) \leq u(\vec{x}) - v(\vec{x}) \leq \max_{\vec{x}_s \in \partial\Omega} (h(\vec{x}_s) - g(\vec{x}_s)) < \varepsilon$$

or

$$\boxed{|u(\vec{x}) - v(\vec{x})| < \varepsilon, \quad \vec{x} \in \Omega \cup \partial\Omega.}$$