Problems to discuss in Section 1.5: 1.5.2, 1.5.3, and 1.5.12

2.3 Method of Separation of Variables.

Consider the IVBVP

$$\begin{cases} u_t = K u_{xx}, & 0<x<L,\ t>0 \qquad (1) \\ u(0,t) = 0, & u(L,t) = 0 \qquad (2) \\ u(x,0) = f(x) & \qquad (3) \end{cases}$$

BC's homogeneous.

look for Solns. in the form:

$$\boxed{u(x,t) = \phi(x)\, G(t).} \qquad (4)$$

Subst. of (4) into (1) leads to

$$\phi(x) \frac{dG}{dt}(t) = K \frac{d^2\phi}{dx^2}(x)\, G(t)$$

Divid. by $K\phi(x)G(t)$

$$\frac{1}{KG(t)} \frac{dG}{dt}(t) = \frac{1}{\phi(x)} \frac{d^2\phi}{dx^2}(x), \quad \text{for all } x \in [0,L] \text{ and } t>0.$$

Only possibility

$$\frac{1}{KG(t)} \frac{dG}{dt}(t) = \frac{1}{\phi(x)} \frac{d^2\phi}{dx^2}(x) = -\lambda.$$

$\underline{\lambda}$ is an arbitrary and $\underline{\text{unknown constant}}$.

From the previous expression, we obtain

$$\boxed{\begin{aligned} \frac{d^2\phi(x)}{dx^2} &= -\lambda\phi(x) \qquad (5) \\ \frac{dG(t)}{dt} &= -\lambda k G(t). \qquad (6) \end{aligned}}$$

Using BC's.

$$\left.\begin{aligned} u(0,t) &= \phi(0)\,G(t) = 0 \\ u(L,t) &= \phi(L)\,G(t) = 0 \end{aligned}\right\} \xrightarrow[\text{trivial solns.}]{\text{To avoid } G(t)\equiv 0} \begin{aligned} \phi(0) &= 0 \\ \phi(L) &= 0 \end{aligned}$$

It results an ODE BVP.

$$\begin{cases} \dfrac{d^2\phi}{dx^2} + \lambda\phi(x) = 0 & (7) \\ \phi(0) = 0, \quad \phi(L) = 0 & (8) \end{cases}$$

It's called an eigenvalue problem. Clearly,

$\phi(x) \equiv 0$ is always a soln. of (7)–(8) for any λ.

However, we are interested in nontrivial solutions.

It can be proved that only for certain λ, there are nontrivial solns.

These special λ's are called eigenvalues of the eigenvalue problem (7)-(8) and the corresponding non-trivial solns. are called eigenfunctions associated to the eigenvalue λ.

We won't require our product soln. to satisfy the I.C. $u(x,0) = f(x)$. Thus, for $G(t)$ we only have the equation

$$\frac{dG}{dt} = -\lambda k G(t)$$

that has as a general soln.

$$\boxed{G(t) = c\, e^{-\lambda k t}} \qquad (9)$$

Discuss sign of λ from physical expectations

Back to solving eigenvalue problem (7)-(8).

$$\begin{cases} \phi''(x) + \lambda\phi = 0 \\ \phi(0) = 0, \quad \phi(L) = 0 \end{cases}$$

Find "λ" and corresponding non-trivial solns.

(I) If $\lambda > 0$ Look for solns. as $\phi(x) = e^{rx}$.

Charact. equ.: $r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{-\lambda} = \pm i\sqrt{\lambda}$.

then

$$\phi(x) = C_1 e^{i\sqrt{\lambda}x} + C_2 e^{-i\sqrt{\lambda}x}$$ Complex soln.

Real General soln.

$$\phi(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x).$$

Use BC's.

$$0 = \phi(0) = C_1$$

$$0 = \phi(L) = C_2 \sin(\sqrt{\lambda}L)$$

For nontrivial solns. $C_2 \neq 0$ and $\sin(\sqrt{\lambda}L) = 0$

$$\Rightarrow \sqrt{\lambda}L = n\pi$$

$$\Rightarrow \boxed{\lambda_n = \frac{(n\pi)^2}{L^2}, \quad n = 1, 2, \ldots}$$

and corresponding eigenfns. are

$$\boxed{\phi_n(x) = C_2 \sin\frac{n\pi}{L}x.} \quad n = 1, 2, \ldots$$

It's possible to prove that for $\lambda = 0$ and $\lambda < 0$ the eigenvalue problem (7)-(8) only has the trivial Solution. (See book pages 44-46.)

<u>Product solns.</u>

So far we have obtained product solutions

$$u_\lambda(x,t) = B_\lambda \sin(\sqrt{\lambda}\, x)\, e^{-\lambda k t}$$

or

$$\boxed{u_n(x,t) = B_n \sin\left(\frac{n\pi}{L} x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}} \qquad (12)$$

$n = 1, 2, \ldots$

These product Solns. satisfy the heat equation (1) and the boundary conditions (2) and (3). (Prove this!). However, the initial condition: $u(x,0) = f(x)$ is satisfied only if $f(x) = u(x,0) = B_n \sin\left(\frac{n\pi}{L} x\right)$. This won't be the case in general.

Principle of Superposition:

Consider the IBVP.

$$\begin{cases} u_t = k\, u_{xx}, & 0<x<L,\ t>0 \\ u(0,t)=0, \quad u(L,t)=0, & t>0 \\ u(0,x) = 4\sin\left(\frac{3\pi}{L}x\right) + 7\sin\left(\frac{8\pi}{L}x\right) \end{cases} \qquad (*)$$

Due to the linearity of the problem, the solution of this IBVP is given by

$$\boxed{u(x,t) = 4\sin\left(\frac{3\pi}{L}x\right) e^{-k\left(\frac{3\pi}{L}\right)^2 t} + 7\sin\left(\frac{8\pi}{L}x\right) e^{-k(8\pi/L)^2 t}} \qquad (13)$$

(Prove that this is a soln)

We will prove soon that IBVP (*) has a unique solution, so (13) is this unique solution.

We will say that $u(x,t)$ is a <u>superposition of two solution</u> of equation (1) satisfying also the homogeneous Dirichlet BC's.

We can extend this principle of superposition to a finite number of product solutions.

It means

$$\boxed{u(x,t) = \sum_{n=1}^{M} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k(n\pi/L)^2 t}} \qquad (14)$$

is also a solution of (1) and BC's (2) and (3).
(prove this!).

However, (14) won't solve an IBVP if $f(x)$ is an arbitrary function different than

$$\boxed{\sum_{n=1}^{M} B_n \sin\left(\frac{n\pi}{L}x\right)}. \qquad (15)$$

What can be proved (chapter 3 in book) is that

1.- If $f(x)$ satisfies certain smoothness conditions to be specified in chapter 3, then $f(x)$ can be approximated (in some sense) by a finite linear comb. of $\sin\left(\frac{n\pi}{L}x\right)$ as the one in (15).

2.- The approximation improves as M is increased.

3.- In the limit $M \to \infty$ (15) will converge to $f(x)$ (in some sense).

Therefore, for $f(x)$ Satisfying properties that are usually found in many physically interesting I.C.

$$\boxed{f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x.} \qquad (15')$$

We also claim (to be proved later) that

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L} x\right) e^{-k\left(n\pi/L\right)^2 t}} \qquad (16)$$

is the Solution of our original IBVP (1) - (3)
(for many $f(x)$ satisfying special conditions to be discussed in chapter 3)

Assuming $u(x,t)$ defined by the infinite series (16) is a convergent series and it's actually a soln. of IBVP (1)-(3) we will now determine the coefficients $\underline{\underline{B_n}}$.

<u>ORTHOGONALITY</u>. Given two functions $f(x)$ and $g(x)$ whose product: $f(x)g(x)$ is integrable on the interval $[a,b]$

If $\int_a^b f(x)g(x)\,dx = 0$, then we will say that $f(x)$ and $g(x)$ are orthogonal.

<u>Example:</u> $$I = \int_0^L \sin\left(\frac{2\pi}{L}x\right) \sin\left(\frac{3\pi}{L}x\right) dx = 0.$$

<u>Proof.-</u>

$$\cos\left(\frac{2\pi}{L}x - \frac{3\pi}{L}x\right) - \cos\left(\frac{2\pi}{L}x + \frac{3\pi}{L}x\right) =$$

$$\cancel{\cos\frac{2\pi}{L}x \cos\frac{3\pi}{L}x} + \sin\frac{2\pi}{L}x \sin\frac{3\pi}{L}x +$$

$$\cancel{-\cos\frac{2\pi}{L}x \cos\frac{3\pi}{L}x} + \sin\frac{2\pi}{L}x \sin\frac{3\pi}{L}x$$

$$= 2 \sin\frac{2\pi}{L}x \sin\frac{3\pi}{L}x$$

Then, $$I = \int_0^L \cos\left(\frac{2\pi}{L}x - \frac{3\pi}{L}x\right) dx - \int_0^L \cos\left(\frac{2\pi}{L}x + \frac{3\pi}{L}x\right) dx$$

$$= \int_0^L \cos\left(-\frac{\pi}{L}x\right) dx - \int_0^L \cos\left(\frac{5\pi}{L}x\right) dx =$$

$$= \left.\frac{\sin(\pi/L\, x)}{\pi/L}\right|_0^L - \left.\frac{\sin(5\pi/L\, x)}{5\pi/L}\right|_0^L = 0.$$

In general, $$I = \int_0^L \sin\frac{n\pi}{L}x \sin\frac{m\pi}{L}x \, dx = 0, \quad n, m > 0, \; n \neq m.$$

The proof is very similar to the one just provided for $n=2$, $m=3$.

To determine the coefficients B_n, we use equation (15').

Multiplying by $\sin \frac{m\pi}{L} x$ both sides and integrating $\int_0^L dx$

$$\int_0^L f(x) \sin \frac{m\pi}{L} x \, dx = \int_0^L \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x \sin \frac{m\pi}{L} x \, dx$$

Assuming $\int$ and Σ can be interchanged

$$= \sum_{n=1}^{\infty} B_n \int_0^L \sin \frac{n\pi}{L} x \sin \frac{m\pi}{L} x \, dx =$$

$$= B_n \int_0^L \sin^2 \frac{n\pi}{L} x \, dx$$

Because $\int_0^L \sin \frac{n\pi}{L} x \sin \frac{m\pi}{L} x \, dx = 0, \quad m \neq n.$ orthog.

Now, $\int_0^L \sin^2\left(\frac{n\pi}{L} x\right) dx \overset{\text{Rmk below}}{=} \frac{L}{2}$

$$\Rightarrow \quad \boxed{B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx.}$$

Rmk: $a = \frac{n\pi}{L}$

$\sin^2 ax + \cos^2 ax = 1$ and $\cos(2ax) = \cos^2(ax) - \sin^2(ax)$

$\Rightarrow \cos(2ax) = 1 - 2\sin^2(ax)$

$\Rightarrow \sin^2(ax) = \frac{1 - \cos(2ax)}{2}$

$$\int_0^L \sin^2 ax \, dx = \int_0^L \frac{1 - \cos(2ax)}{2} dx = \int_0^L \frac{1}{2} dx - \frac{1}{2}\int_0^L \cos(2ax) \, dx = \frac{L}{2} - \frac{1}{2} \frac{\sin(2ax)}{2a}\Big|_0^L$$

$$= \frac{L}{2} - \frac{1}{2} \frac{\sin[2\frac{n\pi}{L}x]}{2(\frac{n\pi}{L})}\Big|_0^L = L/2.$$

Summarizing, for $f(x)$ (Init. cond.) "well behaved" the unique Soln of IBVP (1)-(3) can be represented by the infinite Series

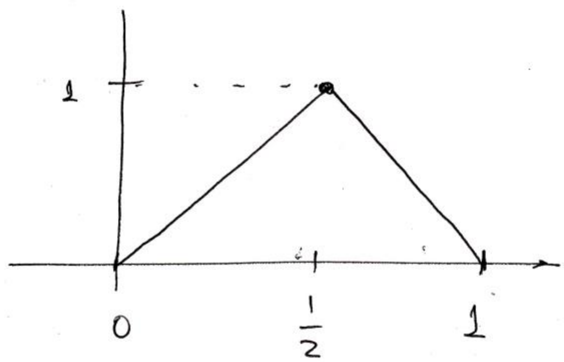
$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x \, e^{-k(n\pi/L)^2 t}$$

where $$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx.$$

Example:

$$\text{IBVP} \begin{cases} u_t = u_{xx}, & 0 < x < 1 \qquad (8.1) \\ u(x,0) = \phi(x) = \begin{cases} 2x, & 0 \le x \le 1/2 \\ 2(1-x), & 1/2 \le x \le 1 \end{cases} & (8.2) \\ u(0,t) = 0, \quad u(1,t) = 0 & (8.3) \end{cases}$$

I.C:



According to our Sep. of Variables procedure. the soln of (8.1) – (8.3) is given by

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\, e^{-\overset{1}{\cancel{k}}(n\pi)^2 t}$$

Where

$$B_n = \frac{2}{1} \int_0^1 \phi(x) \sin(n\pi x)\, dx$$

In our particular case,

$$B_n = 2\left[\int_0^{1/2} 2x \sin(n\pi x)dx + \int_{1/2}^{1} 2(1-x)\sin(n\pi x)dx\right]$$

Knowing that

turn *

$$\int_0^{1/2} x \sin(n\pi x)dx = \frac{\sin\left(\frac{n\pi}{2}\right) - \frac{n\pi}{2}\cos(n\pi/2)}{n^2\pi^2} = A$$

and $$\int_{1/2}^{1} (1-x)\sin(n\pi x)dx = \frac{\sin\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2}\cos\left(\frac{n\pi}{2}\right)}{n^2\pi^2} = B$$

$$\Rightarrow \quad B_n = 4(A+B) = \frac{4\left(2\sin\left(\frac{n\pi}{2}\right)\right)}{n^2\pi^2} = \frac{8\sin\frac{n\pi}{2}}{n^2\pi^2}.$$

$$\Rightarrow \quad \boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{8\sin(n\pi/2)}{n^2\pi^2} e^{-n^2\pi^2 t} \sin(n\pi x).}$$

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} (-1)^{l}, & n = 2l+1, \quad l = 0,1,2. \\ 0, & n = 2l, \quad l = 0,1,2\ldots \end{cases}$$

$$\therefore \quad u(x,t) = \sum_{l=0}^{\infty} \frac{8(-1)^l}{(2l+1)^2\pi^2} e^{-[(2l+1)^2\pi^2]t} \sin\left((2l+1)\pi x\right)$$

$$\int_0^{1/2} x \sin(n\pi x)\,dx = -\frac{x}{n\pi}\cos(n\pi x)\Big|_0^{1/2} + \frac{1}{n^2\pi^2}\sin(n\pi x)\Big|_0^{1/2}$$

$$= \frac{-1}{2n\pi}\cos(n\pi/2) + \frac{1}{n^2\pi^2}\sin(n\pi/2).$$

$$= \frac{-\frac{n\pi}{2}\cos(n\pi/2) + \sin(n\pi/2)}{n^2\pi^2}$$

$$\int_0^{1/2} (1-x)\sin(n\pi x)\,dx = -\frac{(1-x)}{n\pi}\cos(n\pi x)\Big|_{1/2}^{1} - \frac{1}{n^2\pi^2}\sin(n\pi x)\Big|_{1/2}^{1} =$$

$$= \frac{1}{2n\pi}\cos(n\pi/2) + \frac{1}{n^2\pi^2}\sin(n\pi/2) =$$

$$= \frac{\frac{n\pi}{2}\cos(n\pi/2) + \sin(n\pi/2)}{n^2\pi^2}$$