

Semi-infinite String and reflections (II)

12.4.1 (More general)

$$\begin{cases} U_t = c^2 U_{xx}, & x > 0, t > 0 & (1) \\ U(x, 0) = f(x), & x > 0 & (2) \\ U_t(x, 0) = g(x), & x > 0 & (3) \\ U(0, t) = h(t), & t > 0 & (4) \end{cases}$$

General soln. of (1): $U(x, t) = F(x - ct) + G(x + ct)$ (6.1)

Using IC's ^{it} is possible to determine F and G for $x > 0$.

In fact,

$$f(x) = U(x, 0) = F(x) + G(x), \quad x > 0 \quad (5)$$

$$g(x) = U_t(x, 0) = -cF'(x) + cG'(x), \quad x > 0 \quad (6)$$

$$\Rightarrow f'(x) = F'(x) + G'(x)$$

$$\Rightarrow f'(x) + \frac{1}{c} g(x) = 2G'(x) \Rightarrow$$

$$G(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(\bar{x}) d\bar{x} + K$$

$$\text{From (5)} \quad F(x) = f(x) - G(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(\bar{x}) d\bar{x} - K.$$

Summarizing,

$$F(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(\bar{x}) d\bar{x} - K, \quad x > 0 \quad (5.2)$$

$$G(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(\bar{x}) d\bar{x} + K, \quad x > 0 \quad (5.3)$$

Therefore, the soln. of (1)-(4) is completely known

when (I) $x - ct > 0$

$$\begin{aligned} u(x,t) &= F(x-ct) + G(x+ct) = \\ &= \frac{1}{2} [f(x-ct) + f(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\bar{x}) d\bar{x} \quad (7) \end{aligned}$$

However, when

(II) $x - ct < 0$

$F(x-ct)$ is not defined. **yet!** because $f(x-ct)$ is not defined for negative argument.

To obtain the soln. in this region, we use (5.1)

and (4).

$$h(t) = u(0,t) = F(-ct) + G(ct) \quad (8)$$

$F(-ct)$ is not defined. However, from (8)

$$F(-ct) = h(t) - G(ct)$$

if $z = -ct$ then, $ct = -z$, and $t = -\frac{z}{c}$

Thus, $F(z) = h\left(-\frac{z}{c}\right) - G\left(-\frac{z}{c}\right), \quad z < 0 \quad (9)$

$\Rightarrow F(x-ct) = h\left(\frac{ct-x}{c}\right) - G(ct-x).$

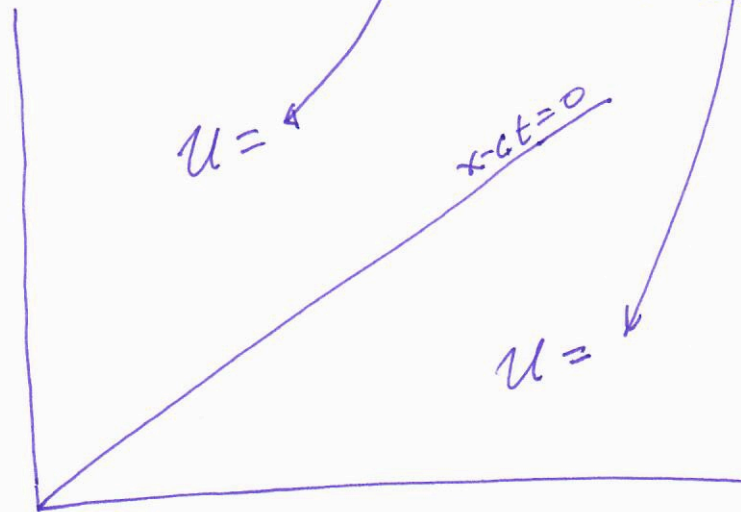
Equ. (9) represents the def. of F for negative numbers

therefore, the solution $u(x,t)$ when $x - ct < 0$ can be written as

$$\begin{aligned} u(x,t) &= F(x-ct) + G(x+ct) = h\left(\frac{ct-x}{c}\right) - \frac{1}{2} f(ct-x) - \frac{1}{2c} \int_0^{ct-x} g(\bar{x}) d\bar{x} \\ &\quad + \frac{1}{2} f(x+ct) + \frac{1}{2c} \int_{ct-x}^{x+ct} g(\bar{x}) d\bar{x} \\ u(x,t) &= \frac{1}{2} [f(x+ct) - f(ct-x)] + \frac{1}{2c} \int_{ct-x}^{x+ct} g(\bar{x}) d\bar{x} + h\left(\frac{ct-x}{c}\right) \quad (10) \end{aligned}$$

Therefore, Solution is given by

$$u(x,t) = \begin{cases} \frac{1}{2} [f(x-ct) + f(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\bar{x}) d\bar{x}, & x-ct > 0. \\ \frac{1}{2} [f(x+ct) - f(ct-x)] + \frac{1}{2c} \int_{ct-x}^{x+ct} g(\bar{x}) d\bar{x}, & x+ct < 0 \\ & + h\left(t - \frac{x}{c}\right) \end{cases}$$



12.4 Semi-Infinite String and Reflections.

$$\left\{ \begin{array}{l} U_{tt} = C^2 U_{xx}, \quad t > 0, \quad x > 0 \\ U(x, 0) = \begin{cases} 1, & 4 < x < 5 \\ 0, & x < 4 \text{ or } x > 5 \text{ and } x > 0. \end{cases} \\ U_t(x, 0) = 0 \\ U(0, t) = 0 \end{array} \right\} \begin{array}{l} \text{IC's.} \\ \longrightarrow \\ \text{B.C.} \end{array}$$

Starting point:

General soln. : $U(x, t) = F(x-ct) + G(x+ct)$
for any (x, t) in $\{(x, t) : x > 0, t > 0\}$.

Also, if I.C. are given by

$$U(x, 0) = f(x) \text{ and } U_t(x, 0) = g(x),$$

General Case

$x > 0$

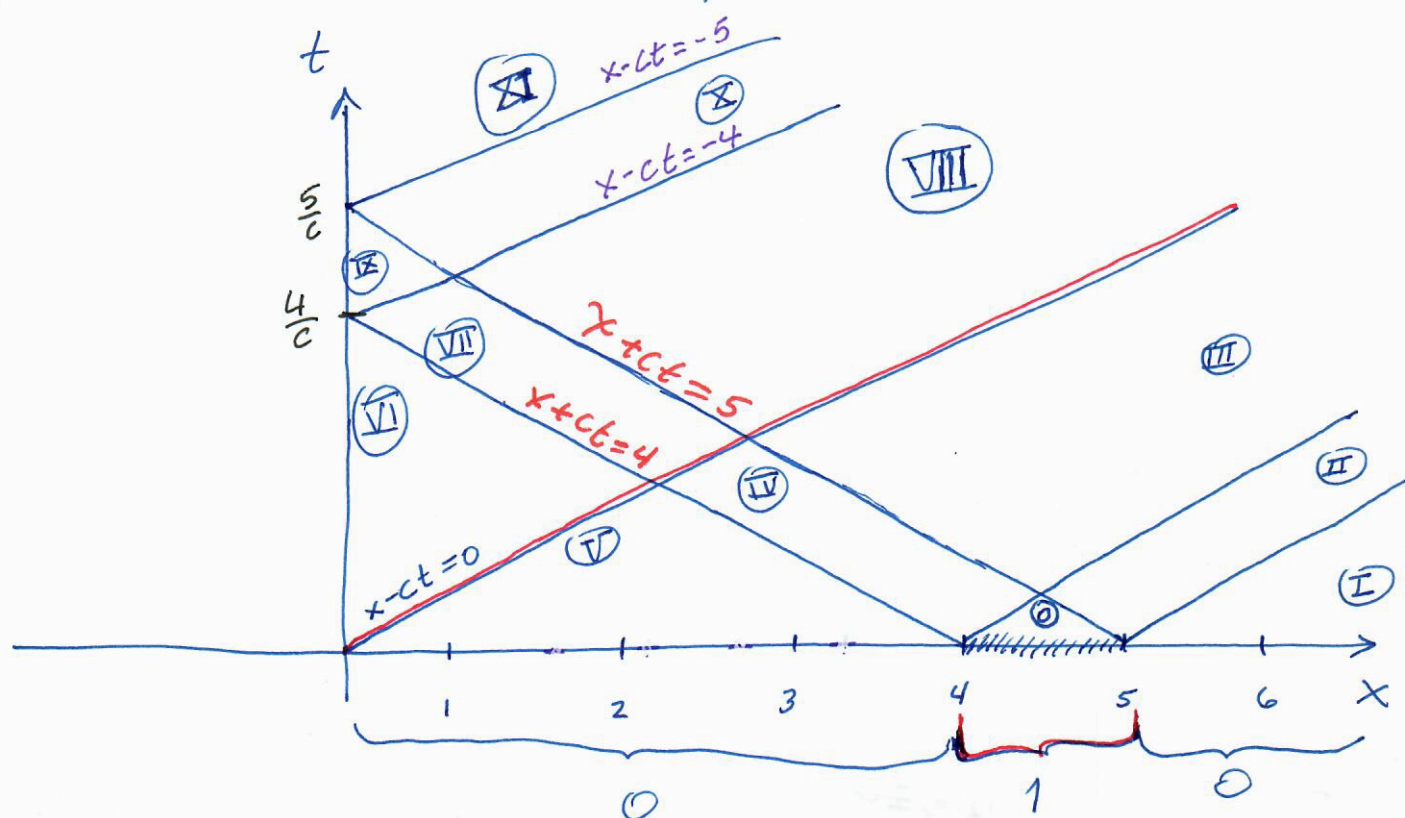
then, $F(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(\bar{x}) d\bar{x}, \quad x > 0$

$$G(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(\bar{x}) d\bar{x}, \quad x > 0$$

In our case,

$$F(x) = G(x) = \begin{cases} \frac{1}{2}, & 4 < x < 5 \\ 0, & x > 5, \quad x < 4 \text{ and } x > 0. \end{cases}$$

Let's consider the x - t plane and the characteristics.



In regions: (I) - (V) Same as in Infinite String.
 No boundary effect below $x-ct=0$

$$u(x,t) = \frac{1}{2} [f(x+ct) + f(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\bar{x}) d\bar{x} \quad \text{Turn} \rightarrow$$

All Characteristics are located in the region $x-ct > 0$ below the line $x-ct=0$.

At regions: (I), (V), and (III) $u(x,t) \equiv \frac{1}{2} [0+0] = 0$

At regions: (IV) and (II), $u(x,t) \equiv \frac{1}{2} [1+0] = \frac{1}{2}$

At region (6): $u(x,t) \equiv \frac{1}{2} [1+1] = 1$.

In all regions: $\textcircled{\text{VI}} - \textcircled{\text{XI}}$

$$\underline{x-ct < 0} \quad \text{and} \quad x+ct > 0.$$

$$\text{and } u(x,t) = F(x-ct) + G(x+ct).$$

The problem is that $F(x)$ is only defined for $x > 0$.

$$\text{and } x-ct < 0$$

Thus, we need to find $F(x)$ for $x < 0$

In order to do this, we use the B.C. at $x=0$.

$$\text{Since } u(0,t) = 0 \Rightarrow F(-ct) + G(ct) = 0$$

$$\Rightarrow F(\overset{0}{-ct}) = -G(\overset{0}{ct})$$

$$\text{Calling } \boxed{y = -ct} \Rightarrow \boxed{t = -y/c}$$

$$F(y) = -G(-y), \quad \text{for any } y < 0.$$

Therefore, the general solution on this part of the domain is given by

$$u(x,t) = F(x-ct) + G(x+ct) = -G(\overset{0}{ct-x}) + G(\overset{0}{x+ct}), \quad \text{when } x-ct < 0$$

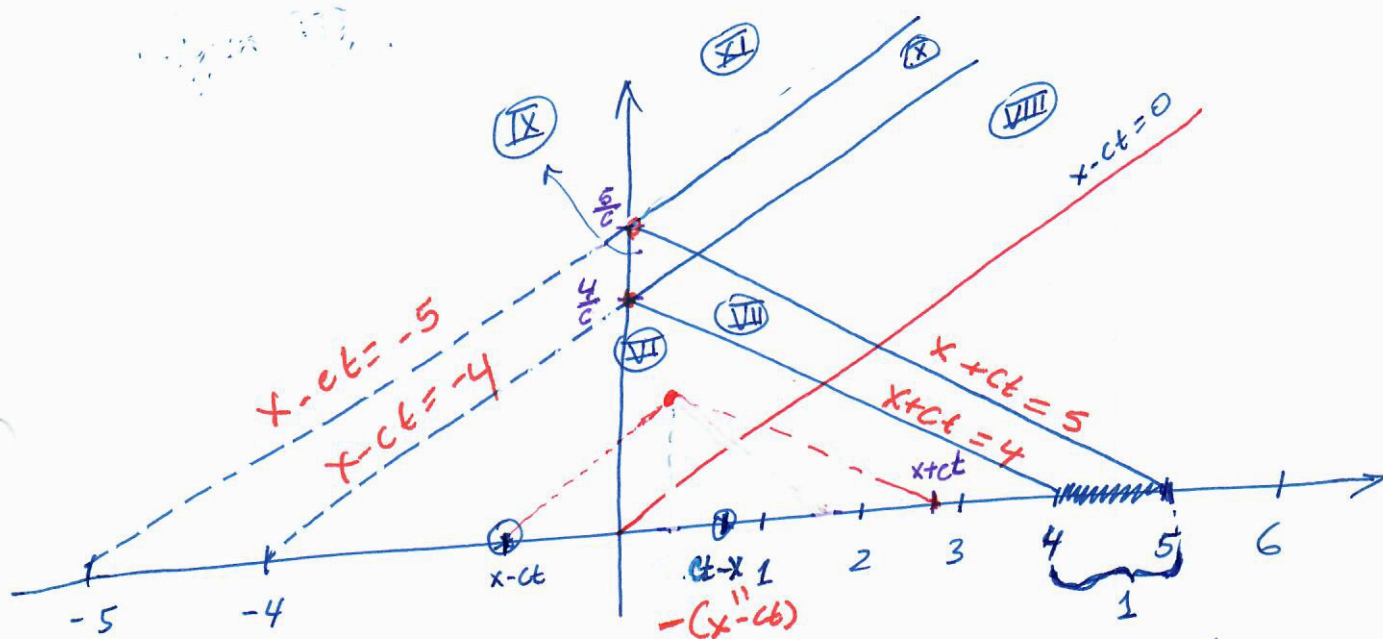
Using the definition of $G(x)$, $x > 0$, for I.C's: $u(x,0) = f(x)$, $u_t(x,0) = g(x)$.

$$\boxed{u(x,t) = \frac{1}{2} [f(x+ct) - f(ct-x)] + \frac{1}{2c} \int_{ct-x}^{x+ct} g(\bar{x}) d\bar{x}} \quad \text{when } x-ct < 0$$

In our particular case $g(x) \equiv 0$

and $u(x,t) = \frac{1}{2} [f(x+ct) - f(ct-x)]$, $x-ct < 0$

Solution at the different regions in the plane



Inside Region (VI): $-4 < x-ct < 0 \Rightarrow 0 < ct-x < 4$
and $0 < x+ct < 4$

$$\Rightarrow u(x,t) = \frac{1}{2} [0+0] = 0$$

Inside Region (VII): $-4 < x-ct < 0 \Rightarrow 0 < ct-x < 4$
and $4 < x+ct < 5$

$$\Rightarrow u(x,t) = \frac{1}{2} [1+0] = \frac{1}{2}$$

Inside Region (IX): $-5 < x-ct < -4 \Rightarrow 4 < ct-x < 5$
and $4 < x+ct < 5$

$$\Rightarrow u(x,t) = \frac{1}{2} \left[\frac{1}{2} - \frac{1}{2} \right] = 0$$

Inside region (IX): $-5 < x-ct < -4 \Rightarrow 4 < ct-x < 5$
and $x+ct > 5$

$$\Rightarrow u(x,t) = \frac{1}{2} [0-1] = -\frac{1}{2}.$$

Inside region (VIII): $-4 < x-ct < 0 \Rightarrow 0 < ct-x < 4$
and $x+ct > 5$

$$\Rightarrow u(x,t) = \frac{1}{2} [0-0] = 0.$$

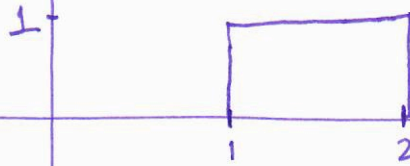
Inside region (XI): $x-ct < -5 \Rightarrow ct-x > 5$
and $x+ct > 5$

$$\Rightarrow u(x,t) = \frac{1}{2} [0-0] = 0.$$

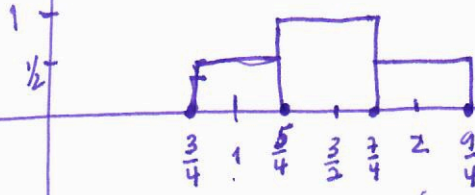
$$u(x,t) = \frac{1}{2} [f(x-ct) - f(ct-x)] \begin{cases} 0, & 0 < x+ct < 4, & -4 < x-ct < 0 & 0 < ct-x < 4 & \text{(VI)} \\ \frac{1}{2}, & 4 < x+ct < 5, & -4 < x-ct < 0 & & \text{(VII)} \\ 0, & x+ct > 5, & -4 < x-ct < 0 & & \text{(VIII)} \\ 0, & 4 < x+ct < 5, & -5 < x-ct < -4 & & \text{(IX)} \\ -\frac{1}{2}, & x+ct > 5, & -5 < x-ct < -4 & & \text{(X)} \\ 0, & x+ct > 5, & x-ct < -5 & & \text{(XI)} \end{cases}$$

(6)

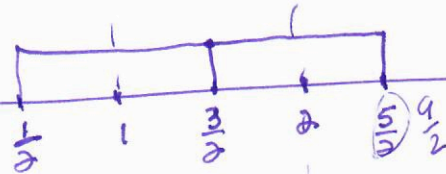
$$t=0$$



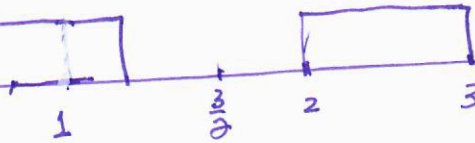
$$t = \frac{1}{4c}$$



$$t = \frac{1}{2c}$$

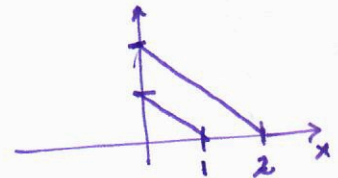
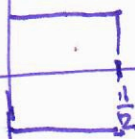


$$t = \frac{1}{c}$$



$$t = \frac{1+t}{2c}$$

$$t = \frac{3}{2c}$$



$$X - ct = 1$$

$$t = \frac{1}{4c}$$

$$\Rightarrow X - \frac{1}{4} = 1$$

$$X = \frac{5}{4}$$

$$X + ct = 2$$

$$t = \frac{1}{4c}$$

$$\Rightarrow X = 2 - \frac{1}{4} = \frac{7}{4}$$

$$\text{At } t = \frac{3}{2c}$$

$$X - ct = 1$$

$$X - c\left(\frac{3}{2c}\right) = 1 \Rightarrow X = \frac{5}{2}$$

$$X + \frac{3}{2} = 1 \Rightarrow X = -\frac{1}{2}$$

$$X + \frac{3}{2} = 2 \Rightarrow X = \frac{1}{2}$$

$$t = \frac{2}{c}$$

