12.3 Summarizing pages ①–④.

$$\begin{cases} u_{tt} = c^2 u_{xx} & -\infty < x < \infty, \quad t > 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

Define:

$$\xi = \xi(x,t) = x - ct, \qquad \eta = \eta(x,t) = x + ct.$$

Inverse: $x = x(\xi,\eta) = \dfrac{\xi+\eta}{2}, \qquad t = t(\xi,\eta) = \dfrac{\eta-\xi}{2c}$

Define:

$$\hat{u}(\xi,\eta) = u(x(\xi,\eta), t(\xi,\eta))$$

Inverse:

$$u(x,t) = \hat{u}(\xi(x,t), \eta(x,t))$$

We prove that

$$u_{tt} - c^2 u_{xx} = 0 \quad \Rightarrow \quad \hat{u}_{\xi\eta}(\xi,\eta) = 0.$$

Integrating once w.r.t. η

$$\hat{u}_\xi(\xi,\eta) = f(\xi)$$

Integrating now w.r.t. ξ

$$\hat{u}(\xi,\eta) = \int f(\xi)\,d\xi + G(\eta) = F(\xi) + G(\eta).$$

$$\Rightarrow \quad u(x,t) = \hat{u}(\xi(x,t), \eta(x,t)) = F(\xi(x,t)) + G(\eta(x,t)).$$

or $\boxed{u(x,t) = F(x-ct) + G(x+ct)}$ ✓

All Solutions of the wave equation are of the form:

$$u(x,t) = F(x-ct) + G(x+ct) \quad (6.0)$$ F, G arbitrary Twice differentiable.

We will prove that

$$F(x) = \frac{1}{2}\left[f(x) - \frac{1}{c}\int_a^x g(\bar{x})\,d\bar{x}\right] + K \qquad (6.1)$$

and
$$G(x) = \frac{1}{2}\left[f(x) + \frac{1}{c}\int_a^x g(\bar{x})\,d\bar{x}\right] - K. \qquad (6.2)$$

To obtain (6.1) and (6.2) from (6.0), we use the IC's.

$$f(x) = u(x,0) = F(x) + G(x). \qquad (6.3)$$

$$g(x) = u_t(x,0) = -cF'(x) + cG'(x). \qquad (6.4)$$

Taking derivative of (6.3), multiplying it by $(-c)$ and adding to (6.4)

$$-cf'(x) = -cF'(x) - \cancel{cG'(x)}$$

$$g(x) = -cF'(x) + \cancel{cG'(x)}$$

$$-cf'(x) + g(x) = -2cF'(x) \Rightarrow cF'(x) = \frac{1}{2}cf'(x) - \frac{1}{2}g(x)$$

Integrating $\int_a^x dx$

$$cF(x) = \frac{1}{2}\left[cf(x) - \int_a^x g(\bar{x})\,d\bar{x}\right] + \underbrace{cF(a) - \frac{1}{2}cf(a)}_{\tilde{K}}$$

or

$$\boxed{F(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_a^x g(\bar{x})\,d\bar{x} + K.}$$

Substitution into (6.3)

$$G(x) = f(x) - F(x) = f(x) - \frac{1}{2} f(x) + \frac{1}{2c}\int_a^x g(\bar{x})\,d\bar{x} - K$$

$$\Rightarrow \quad \boxed{G(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_a^x g(\bar{x})\,d\bar{x} - K}$$

Therefore,

$$u(x,t) = F(x-ct) + G(x+ct) =$$

$$= \frac{1}{2} f(x-ct) - \frac{1}{2c}\int_a^{x-ct} g(\bar{x})\,d\bar{x} + \cancel{K}$$

$$+ \frac{1}{2} f(x+ct) + \frac{1}{2c}\int_a^{x+ct} g(\bar{x})\,d\bar{x} - \cancel{K}$$

$$\Rightarrow \quad \boxed{u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x}} \qquad (7.1)$$

D'Alembert formula.

First, we will verify that (·.1) is a solution of the wave equ.

$$u_t \overset{\text{Leibniz's rule}}{=} \frac{1}{2}\left[f'(x-ct)(-c) + f'(x+ct)(c)\right] + \frac{1}{2c}\left[g(x+ct)*c - g(x-ct)*(-c)\right]$$

$$\Rightarrow u_{tt} = \frac{1}{2}\left[c^2 f''() + c^2 f''\right] + \frac{1}{2c}\left[c^2 g'() - c^2 g'()\right]$$

Similarly,

$$u_{xx} = \frac{1}{2}\left[f''() + f''() + \frac{1}{c}\left(g'() - g'()\right)\right]$$

Thus,

$u_{tt} - c^2 u_{xx} = 0$ ✓ and (4.1) is a soln. of the wave equation if $f(x)$ is twice differentiable and $g(x)$ has 1st derivative.

Eq. (4.1) also satisfies the I.C's., since

$$u(x,0) = \frac{f(x)+f(x)}{2} + \frac{1}{2c}\underbrace{\int_x^x g(\bar{x})\,dx}_{=0} = f(x) \checkmark$$

$$u_t(x,t) = \frac{1}{2}\left[-cf'(x-ct) + cf'(x+ct)\right] + \frac{1}{2c}\left[g(x+ct)c + c\,g(x-ct)\right]$$

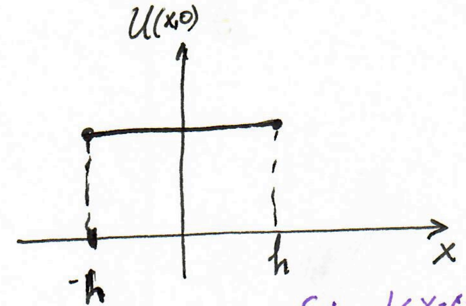
$$\Rightarrow u_t(x,0) = \frac{1}{2}\left[\cancel{-cf'(x) + cf'(x)}\right] + \frac{1}{2c}\left[cg(x) + cg(x)\right] = g(x) \checkmark$$

Leibniz rule:

$$\frac{d}{dx}\left[\int_{f(x)}^{g(x)} h(t,x)\,dt\right] = \int_{f(x)}^{g(x)} \frac{\partial h}{\partial x}(t,x)\,dt + h(t,g(x))\,g'(x) - h(t,f(x))\,f'(x).$$

Example.- Solve the IVP.

$$\begin{cases} u_{tt} = c^2 u_{xx}, & -\infty < x < \infty, \quad t > 0. \\ u(x,0) = f(x) = \begin{cases} 1, & |x| < h \\ 0, & |x| > h \end{cases} \\ u_t(x,0) = g(x) \equiv 0. \end{cases}$$



Notice that $f(x-ct) = \begin{cases} 1, & -h < x-ct < h \\ 0, & \text{otherwise} \end{cases}$

$f(x+ct) = \begin{cases} 1, & -h < x+ct < h \\ 0, & \text{otherwise} \end{cases}$

D'Alambert's Soln.

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right]}$$

In Region I: $-h < x-ct < h$ and $-h < x+ct < h$

$$u(x,t) = \frac{1}{2}\left[1+1\right] = 1.$$

In Region II: $-h < x-ct < h$ and $x+ct > h$.

$$u(x,t) = \frac{1}{2}\left[\overset{=1}{f(x-ct)} + \overset{=0}{f(x+ct)}\right] = \frac{1}{2}.$$

In Region III: $x-ct < -h$ and $|x+ct| < h$.

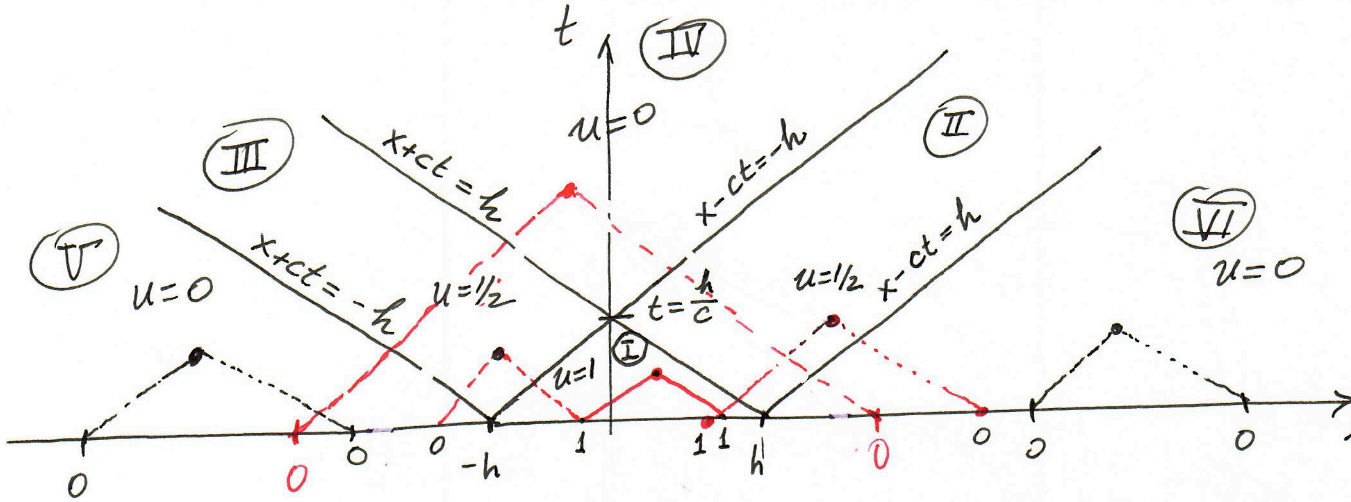
$$u(x,t) = \frac{1}{2}\left[\overset{=0}{f(x-ct)} + \overset{=1}{f(x+ct)}\right] = \frac{1}{2}.$$

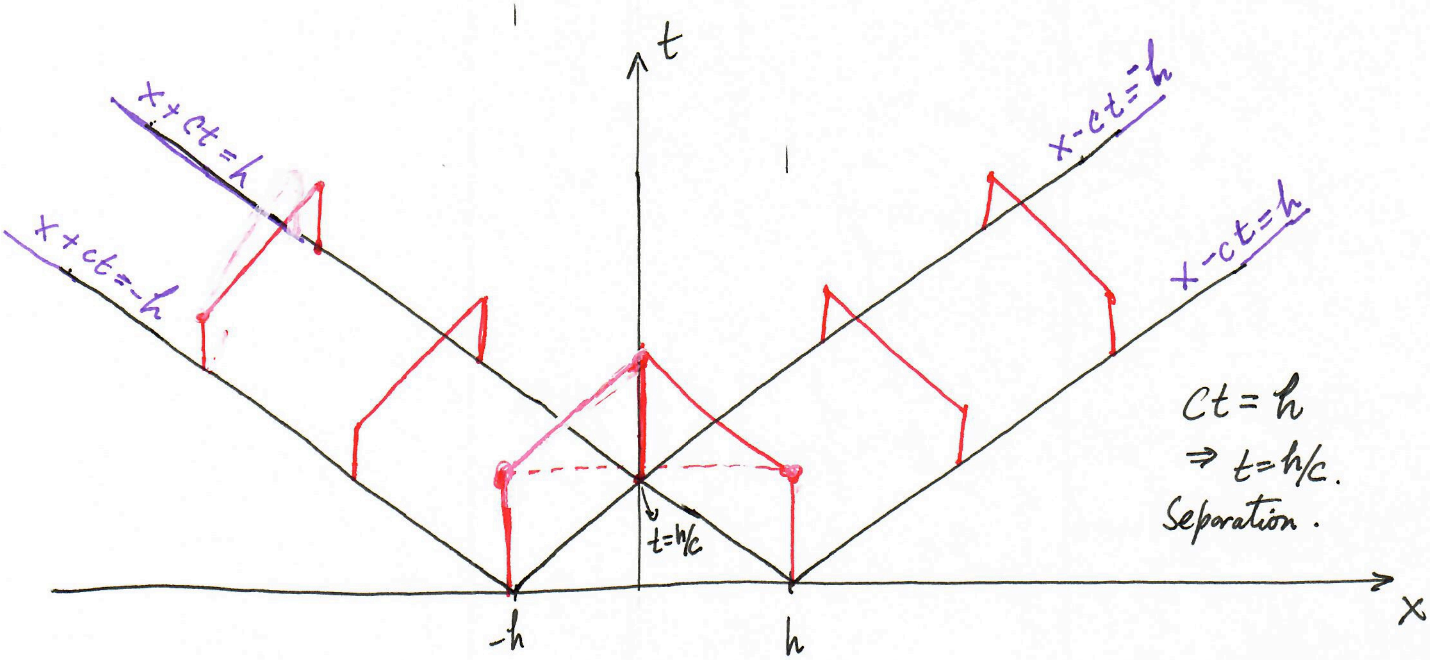
In Region IV: $x-ct < -h$ and $x+ct > h$.

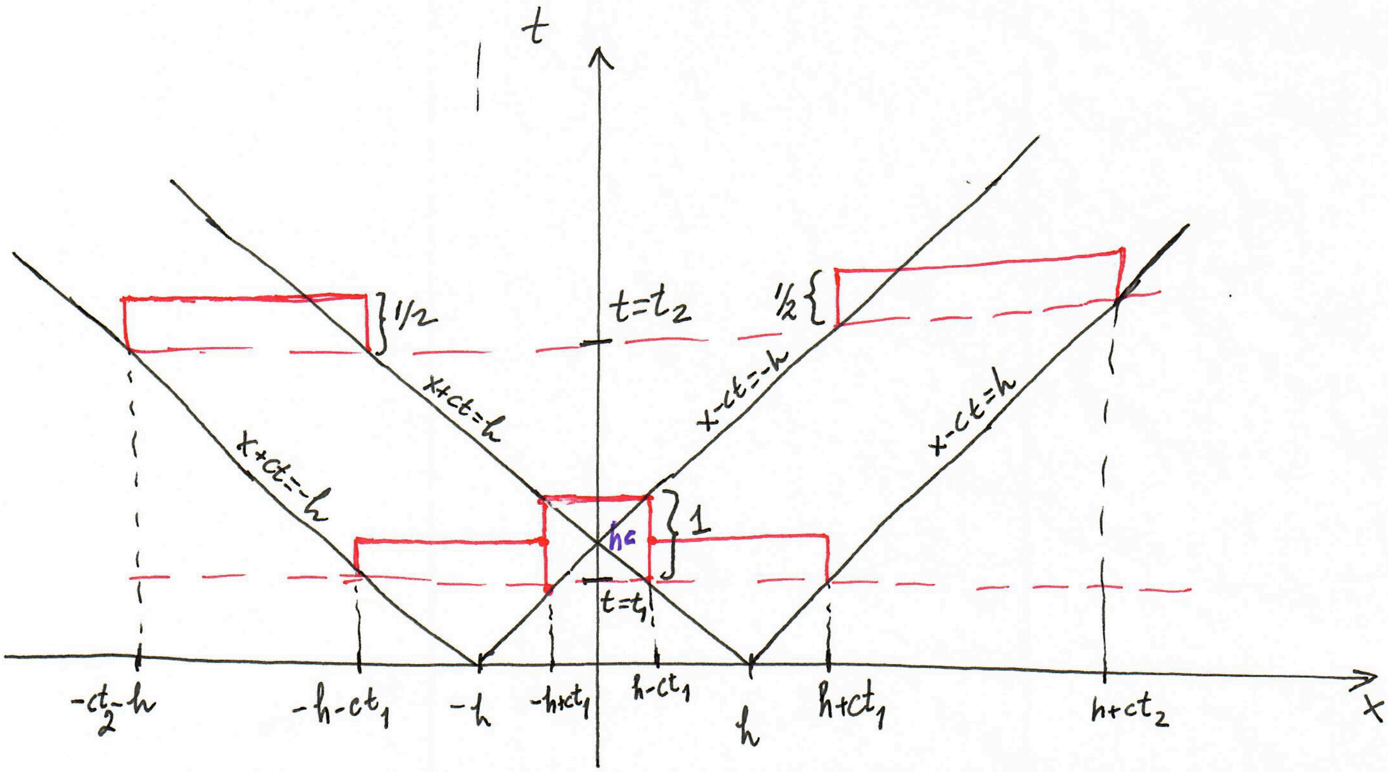
$$u(x,t) = \frac{1}{2}\left[0+0\right] = 0.$$

In Region V: $x-ct < -h$ and $x+ct < -h$. $\Rightarrow u(x,t) = \frac{1}{2}[0+0] = 0$

In Region VI: $x-ct > h$ and $x+ct > h \Rightarrow u(x,t) = \frac{1}{2}[0+0] = 0.$


t
IV
III
II
V
VI
u=0
x+ct=h
x-ct=-h
x+ct=-h
x-ct=h
u=1/2
u=1/2
u=0
t=h/c
u=1
I
-h
h
x


t
x+ct=h
x+ct=-h
x-ct=-h
x-ct=h
ct = h
⇒ t=h/c.
Separation.
t=h/c
-h
h
x


t
t=t2
1/2
1/2
x+ct=h
x-ct=-h
x+ct=-h
x-ct=h
1
h/c
t=t1
-ct2-h
-h-ct1
-h
-h+ct1
h-ct1
h
h+ct1
h+ct2
x