

Green's function for bounded domain using infinite Space Green's function.

$$\begin{cases} \nabla^2 G = \delta(\vec{x} - \vec{x}_0), & \vec{x} \in \Omega \text{ (bounded domain)} \\ + \text{homog BC's.} \end{cases} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

Idea is to use infinite space Green's fn. + something else

$$G(\vec{x}, \vec{x}_0) = \frac{1}{2\pi} \ln |\vec{x} - \vec{x}_0| + v(\vec{x}, \vec{x}_0).$$

Where $v(\vec{x}, \vec{x}_0)$ satisfies

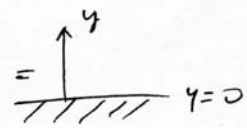
$$\begin{cases} \nabla^2 v = 0, & \vec{x} \in \Omega \\ v(\vec{x}_s, \vec{x}_0) = -\frac{1}{2\pi} \ln |\vec{x}_s - \vec{x}_0|, & \vec{x}_s \in \partial\Omega \end{cases} \quad (3) \quad (4)$$

If boundary is simple (3) and (4) can be solved by Separation of variables. In general, it's not easy to solve (3) and (4).

Method of Images. Semi-infinite plane ($y > 0$).

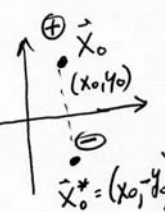
$$\text{Consider } \begin{cases} \nabla^2 u = f(x), & \vec{x} \in \Omega \\ u(\vec{x}_s) = h(\vec{x}_s), & \vec{x}_s \in \partial\Omega = \{(x, 0) \mid y=0\} \end{cases}$$

Corresponding Green's function satisfies:

$$\begin{cases} \nabla^2 G(\vec{x}, \vec{x}_0) = \delta(\vec{x} - \vec{x}_0), & \vec{x} \in \Omega \\ G(x, 0; x_0, y_0) = 0, \end{cases}$$


In this case, there is an easy way to construct $v(\vec{x})$ satisfying (3) and (4).

Consider the infinite space Green's function

$$\nabla^2 \bar{G}(\vec{x}, \vec{x}_0^*) = -\delta(\vec{x} - \vec{x}_0^*), \quad \vec{x} \in \mathbb{R}^2$$


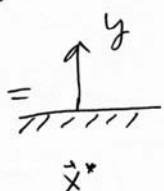
where $\vec{x}_0^* = (x_0, -y_0) \Rightarrow x_0^* = x_0, y_0^* = -y_0$

then, $\boxed{\bar{G}(\vec{x}, \vec{x}_0^*) = -\frac{1}{2\pi} \ln |\vec{x} - \vec{x}_0^*|} \quad (5)$

and at the x -axis = $\{(x, y) \text{ such that } y=0\}$

$$\boxed{\bar{G}(x, 0; x_0^*, y_0^*) = -\frac{1}{2\pi} \ln [(x-x_0)^2 + y_0^2]^{1/2}} \quad (6)$$

Also, $\nabla^2 \bar{G}(\vec{x}, \vec{x}_0^*) = 0, \quad \vec{x} \in \Omega$



because $\vec{x}^*$ is in the lower half

plane. $\vec{x}^* \notin \text{Upper half plane} \Rightarrow \delta(\vec{x} - \vec{x}_0^*) = 0$
for $\vec{x} \in \Omega$.

Therefore, the restriction of $\bar{G}(\vec{x}, \vec{x}_0^*)$ to the half plane ($y > 0$) is the function $v(\vec{x})$ sought, which satisfies (3) and (4) $\vec{x}_0^* = (x_0, -y_0)$

$$\begin{cases} \nabla^2 \bar{G}(\vec{x}, \vec{x}_0^*) = 0, & \{(x, y) \mid y > 0\} = \mathcal{R} \\ \bar{G}(x, 0; \underset{x_0}{x}_0^*, \underset{-y_0}{y}_0^*) = -\frac{1}{2\pi} \ln(\sqrt{(x-x_0)^2 + y_0^2}) \end{cases}$$

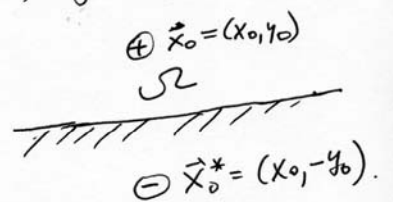
Therefore, the Green's function for the half plane is given by

$$\begin{aligned} G(\vec{x}, \vec{x}_0) &= \frac{1}{2\pi} \ln |\vec{x} - \vec{x}_0| - \frac{1}{2\pi} \ln |\vec{x} - \vec{x}_0^*| = \\ &= \frac{1}{2\pi} \ln \left[\frac{((x-x_0)^2 + (y-y_0)^2)^{1/2}}{((x-x_0)^2 + (y+y_0)^2)^{1/2}} \right] \end{aligned}$$

$$\therefore \boxed{G(\vec{x}, \vec{x}_0) = \frac{1}{4\pi} \ln \left[\frac{(x-x_0)^2 + (y-y_0)^2}{(x-x_0)^2 + (y+y_0)^2} \right]} \quad (7)$$

Verification:

$$\begin{aligned}\nabla^2 G(\vec{x}, \vec{x}_0) &= \nabla^2 \left(\frac{1}{2\pi} \ln |\vec{x} - \vec{x}_0| \right) - \nabla^2 \left(\frac{1}{2\pi} \ln |\vec{x} - \vec{x}_0^*| \right) \\ &= \delta(\vec{x} - \vec{x}_0) - \delta(\vec{x} - \vec{x}_0^*) = \delta(\vec{x} - \vec{x}_0) \\ &\text{for } \vec{x} \in \{ \text{Half plane, } y > 0 \}.\end{aligned}$$



At the boundary $\vec{x}_s = (x, 0)$.

Also, $G(\vec{x}_s, \vec{x}_0) = G(x, 0; x_0, y_0) =$

$$= \frac{1}{4\pi} \ln \left[\frac{(x-x_0)^2 + y_0^2}{(x-x_0)^2 + y_0^2} \right] = \frac{1}{4\pi} \ln(1) = 0 \quad \checkmark$$

Back to Nonhomogeneous problem:

$$\begin{cases} \nabla^2 u = f(x), & + \text{ B.C. } u(x, 0) = h(x). \\ \Omega \equiv \text{Half plane } y > 0. \end{cases}$$

Use Green's formula:

$$\int_{\text{Half plane}} [u \nabla^2 G - G \nabla^2 u] d\vec{s} = \oint_{\text{boundary}} [u \nabla G - G \nabla u] \cdot \hat{n} d\ell$$

or

$$\underbrace{\iint [u \delta(\vec{x}-\vec{x}_0) - G f(\vec{x})] ds_{\vec{x}}}_{\text{hatched}} = \int_{-\infty}^{\infty} \left[u \left(\frac{\partial G}{\partial y} \right) - G \left(-\frac{\partial u}{\partial y} \right) \right] dx$$

$$+ \underbrace{\int_0^{\pi} \left[u \frac{\partial G}{\partial r} - G \frac{\partial u}{\partial r} \right] r d\theta}_{\text{hatched}} \bigg|_{r \rightarrow \infty} \rightarrow 0.$$

Def of
infinite space Green's fn.

$$\Rightarrow \underbrace{u(\vec{x}_0)}_{\text{hatched}} = \iint \underbrace{G(\vec{x}, \vec{x}_0)}_{\text{hatched}} f(\vec{x}) ds_{\vec{x}} - \int_{-\infty}^{\infty} h(\vec{x}) \frac{\partial G}{\partial y}(\vec{x}, \vec{x}_0) \bigg|_{y=0} dx$$

Reversing roles $x \rightarrow x_0$ + Symmetry of G .

$$\underbrace{u(\vec{x})}_{\text{hatched}} = \iint G(\vec{x}, \vec{x}_0) f(\vec{x}_0) ds_{\vec{x}_0} - \int_{-\infty}^{\infty} h(\vec{x}_0) \frac{\partial G}{\partial y_0}(\vec{x}, \vec{x}_0) \bigg|_{y_0=0} dx_0$$

Now, $\frac{\partial G}{\partial y}(\vec{x}, \vec{x}_0) \bigg|_{y_0=0} = \frac{\partial G}{\partial y}(x, y; x_0, 0) = \frac{-y/\pi}{(x-x_0)^2 + y^2}$

In fact,

$$G(\vec{x}, \vec{x}_0) = \frac{1}{4\pi} \left[\ln[(x-x_0)^2 + (y-y_0)^2] - \ln[(x-x_0)^2 + (y+y_0)^2] \right]$$

$$\Rightarrow \frac{\partial}{\partial y_0} G(\vec{x}, \vec{x}_0) = \frac{1}{4\pi} \left[\frac{-2(y-y_0)}{(x-x_0)^2 + (y-y_0)^2} - \frac{2(y+y_0)}{(x-x_0)^2 + (y+y_0)^2} \right]$$

$$\Rightarrow \frac{\partial G}{\partial y_0}(x, y; x_0, 0) = \frac{1}{4\pi} \left[\frac{-4y}{(x-x_0)^2 + y^2} \right]$$

dipole source.

In particular,
the BVP in the half plane given by

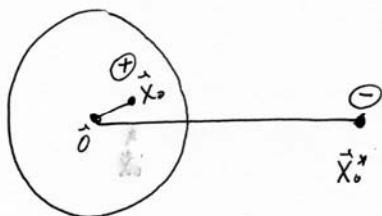
$$\begin{cases} \nabla^2 u = 0 \\ u(x, 0) = h(x) \end{cases}$$

has as a solution:

$$\begin{aligned} u(x, y) &= - \int_{-\infty}^{\infty} \left(h(x_0) \frac{\partial G}{\partial y_0}(\vec{x}, \vec{x}_0) \right) dx_0 = \\ &= \frac{y}{\pi} \int_{-\infty}^{\infty} h(x_0) \frac{1}{(x-x_0)^2 + y^2} dx_0 \end{aligned}$$

Method of Images. Green's function for a circle.

$$\begin{cases} \nabla^2 G(\vec{x}, \vec{x}_0) = \delta(\vec{x} - \vec{x}_0), & \vec{x} \in \text{Circle of radius } a. \\ G(\vec{x}, \vec{x}_0) = 0, & |\vec{x}| = a \end{cases} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$



The idea is to find $\vec{x}_0^*$ ^{somewhere outside the circle} and define $\bar{G}(\vec{x}, \vec{x}_0)$ in the whole plane $\mathbb{R}^2$ satisfying

$$\begin{cases} \nabla^2 \bar{G}(\vec{x}, \vec{x}_0) = \delta(\vec{x} - \vec{x}_0) - \delta(\vec{x} - \vec{x}_0^*), & \vec{x} \in \mathbb{R}^2. \\ \bar{G}(\vec{x}, \vec{x}_0) = 0, & |\vec{x}| = a. \end{cases} \quad \begin{matrix} (3) \\ (4) \end{matrix}$$

This $\bar{G}(\vec{x}, \vec{x}_0)$ restricted to the region bounded by the circumference $r=a$ is the Green's function for the circle with homog. Dirichlet B.C., i.e.,

$$G(\vec{x}, \vec{x}_0) = \bar{G}(\vec{x}, \vec{x}_0) \Big|_{|\vec{x}|=a}$$

The solution of (3) and (4) can be obtained as a superposition of a positive source located at $\vec{x}_0$ and of a negative source located at $\vec{x}_0^*$.

$$G(\vec{x}, \vec{x}_0) = \frac{1}{2\pi} \ln |\vec{x} - \vec{x}_0| - \frac{1}{2\pi} \ln |\vec{x} - \vec{x}_0^*| + C \quad (5)$$

where C is a constant to be determined.

Then, we have two unknowns: $\vec{x}_0^*$ and C .

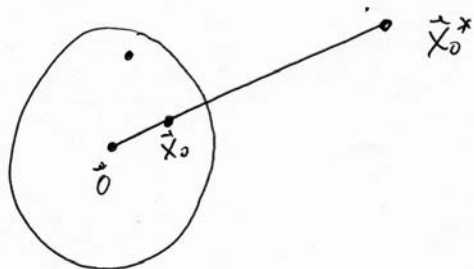
We also have one condition:

$$\left. G(\vec{x}, \vec{x}_0) \right|_{|\vec{x}|=a} = 0 \quad (6)$$

And an additional condition is that

$$\vec{x}_0^* = \mu \vec{x}_0 \quad (7)$$

It means that the image point is along the same radial line as the source point $\vec{x}_0$.



Condition (6) $G|_{r=a} = 0$ is satisfied ^{by (5)} if

$$G(\vec{x}, \vec{x}_0) = \frac{1}{4\pi} \ln \frac{|\vec{x} - \vec{x}_0|^2}{|\vec{x} - \vec{x}_0^*|^2} \Big|_{|\vec{x}|=a} + C = 0$$

$$\Rightarrow C = -\frac{1}{4\pi} \ln \frac{|\vec{x} - \vec{x}_0|^2}{|\vec{x} - \vec{x}_0^*|^2} \Big|_{|\vec{x}|=a} \Rightarrow -4\pi C = \ln \left(\frac{|\vec{x} - \vec{x}_0^*|^2}{|\vec{x} - \vec{x}_0|^2} \right) \Big|_{|\vec{x}|=a}$$

$$\Rightarrow C = e^{-4\pi C} = \frac{|\vec{x} - \vec{x}_0|^2}{|\vec{x} - \vec{x}_0^*|^2} \Big|_{|\vec{x}|=a}$$

$$\Rightarrow \frac{|\vec{x} - \vec{x}_0|^2}{|\vec{x}|=a} = \underbrace{e^{-4\pi C}}_{\parallel K} \frac{|\vec{x} - \vec{x}_0^*|^2}{|\vec{x}|=a}$$

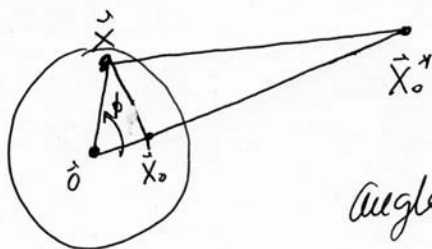
$$\boxed{|\vec{x} - \vec{x}_0|^2 = K |\vec{x} - \vec{x}_0^*|^2} \text{ when } |\vec{x}|=a \quad (8).$$

Now,

$$\begin{aligned} |\vec{x} - \vec{x}_0|^2 &= (\vec{x} - \vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = |\vec{x}|^2 + |\vec{x}_0|^2 - 2 \vec{x} \cdot \vec{x}_0 \\ &= |\vec{x}|^2 + |\vec{x}_0|^2 - 2 |\vec{x}| |\vec{x}_0| \cos \phi. \end{aligned}$$

Similarly,

$$|\vec{x} - \vec{x}_0^*|^2 = |\vec{x}|^2 + |\vec{x}_0^*|^2 - 2 |\vec{x}| |\vec{x}_0^*| \cos \phi.$$



angle betw $\vec{r}$ and $\vec{r}_0$

Same as " betw $\vec{x}$ and $\vec{r}_0$

Over the Circle $|\vec{r}| = a$

$|\vec{r}_0^*| = \mu |\vec{r}_0| = \mu r_0$, where $|\vec{r}_0| = r_0$

and equ. (8) holds. Therefore,

$$a^2 + r_0^2 - 2ar_0 \cos \phi = K(a^2 + \mu^2 r_0^2 - 2a\mu r_0 \cos \phi)$$

true for all ϕ .

$$\Rightarrow \begin{cases} a^2 + r_0^2 = K(a^2 + \mu^2 r_0^2) & (*.1) \quad (\phi = \frac{\pi}{2}) \\ \text{and } 2ar_0 = (2a\mu r_0)K & (*.2) \end{cases}$$

$$\Rightarrow \boxed{\text{From } (*.2) \quad K = 1/\mu}$$

$$\text{Subst. in } (*.1) : a^2 + r_0^2 = \frac{1}{\mu} a^2 + \mu r_0^2$$

$$\Rightarrow \boxed{\mu = a^2/r_0^2}$$

$$\Rightarrow \boxed{\vec{r}_0^* = \frac{a^2}{r_0^2} \vec{r}_0}$$

Location of $\sqrt{\text{negative}}$ point source
outside circle.

$$\Rightarrow k = \frac{r_0^2}{a^2} \Rightarrow k = e^{-4\pi c} \Rightarrow \frac{r_0^2}{a^2} = e^{-4\pi c} \Rightarrow$$

$$\Rightarrow -4\pi c = \ln\left(\frac{r_0^2}{a^2}\right) \Rightarrow c = -\frac{1}{4\pi} \ln\left(\frac{r_0^2}{a^2}\right) =$$

$$\therefore \boxed{c = \frac{1}{4\pi} \ln\left(\frac{a^2}{r_0^2}\right)}$$

Therefore, Subst. in (5) we can obtain the Green's fu. for our problem (Circle of radius a)
Dirichlet cond.

$$\boxed{G(x, \tilde{x}_0) = \frac{1}{4\pi} \ln \frac{|\tilde{x} - x_0|^2}{|\tilde{x} - x_0^*|^2} + \frac{1}{4\pi} \ln\left(\frac{a^2}{r_0^2}\right)}$$

$$\text{where } \tilde{x}_0^* = \frac{a^2}{r_0^2} \tilde{x}_0 \Rightarrow r_0^* = \frac{a^2}{r_0^2} r_0 = \frac{a^2}{r_0}$$

$$\text{or } \boxed{G(\tilde{x}, \tilde{x}_0) = \frac{1}{4\pi} \ln \left(\frac{|\tilde{x} - \tilde{x}_0|^2 a^2}{|\tilde{x} - \tilde{x}_0^*|^2 r_0^2} \right)}$$

$$\begin{aligned} \text{or } G(\tilde{x}, \tilde{x}_0) &= \frac{1}{4\pi} \ln \left(\frac{a^2}{r_0^2} \frac{r^2 + r_0^2 - 2rr_0 \cos \phi}{r^2 + r_0^{*2} - 2rr_0^* \cos \phi} \right) \\ &= \frac{1}{4\pi} \ln \left(\frac{a^2}{r_0^2} \frac{r^2 + r_0^2 - 2rr_0 \cos \phi}{r^2 + \frac{a^4}{r_0^2} - 2r \frac{a^2}{r_0} \cos \phi} \right) \end{aligned}$$

$$\therefore \boxed{G(\vec{x}, \vec{x}_0) = \frac{1}{4\pi} \ln \left(a^2 \frac{r^2 + r_0^2 - 2rr_0 \cos \phi}{r^2 r_0^2 + a^4 - 2rr_0 a^2 \cos \phi} \right)} \quad (9)$$

Then, the solution for the nonhomogeneous BVP.

$$\begin{cases} \nabla^2 u = f(\vec{x}), & \vec{x} \in \text{disk} \\ u(\vec{x}_s) = h(\vec{x}_s), & \vec{x}_s \in \partial\Omega = \text{circle} \end{cases}$$

is given by $u(\vec{x}) = \iint_{\Omega} f(\vec{x}_0) G(\vec{x}, \vec{x}_0) dS_{\vec{x}_0} + \oint_0^{2\pi} h(\vec{x}_0) \nabla_{\vec{x}_0} G(\vec{x}, \vec{x}_0) \cdot \hat{n} d\ell_{\vec{x}_0}$

$$\text{or } \boxed{u(\vec{x}) = \int_0^a \int_0^{2\pi} f(\vec{x}_0) G(\vec{x}, \vec{x}_0) r dr d\phi + \int_0^{2\pi} h(\vec{x}_0) \left. \frac{\partial G}{\partial r_0}(\vec{x}, \vec{x}_0) \right|_{r_0=a} a d\phi} \quad (10)$$

Now,

$$\frac{\partial G}{\partial r_0} = \frac{1}{4\pi} \left(\frac{2r_0 - 2r \cos \phi}{r^2 + r_0^2 - 2rr_0 \cos \phi} - \frac{2r^2 r_0 - 2ra^2 \cos \phi}{r^2 r_0^2 + a^4 - 2rr_0 a^2 \cos \phi} \right)$$

$$\Rightarrow \frac{\partial G}{\partial r_0}(\vec{x}, \vec{x}_0) = \frac{1}{4\pi} \left(\frac{2r_0 - 2r \cos \phi}{r^2 + r_0^2 - 2rr_0 \cos \phi} - \frac{2r^2/r_0 - 2r \frac{a^2}{r_0^2} \cos \phi}{r^2 + \frac{a^4}{r_0^2} - 2r \frac{a^2}{r_0} \cos \phi} \right)$$

$$\Rightarrow \left. \frac{\partial G}{\partial r_0} \right|_{r_0=a} = \frac{1}{4\pi} \left(\frac{2a - 2r \cos \phi - 2r^2/a + 2r \cos \phi}{r^2 + a^2 - 2ra \cos \phi} \right)$$

$$\therefore \left. \frac{\partial G}{\partial r_0} \right|_{r_0=a} = \frac{2a}{4\pi} \left(\frac{1 - (r/a)^2}{r^2 + a^2 - 2ra \cos \phi} \right)$$

Therefore, if our problem is

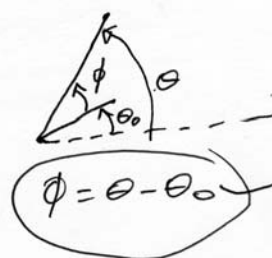
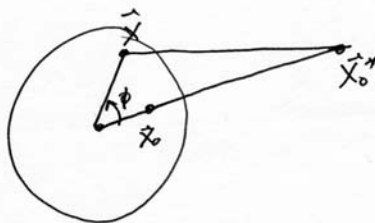
$$\begin{cases} \nabla^2 u = 0, & \vec{x} \in \text{disk} \\ u(\vec{x}_s) = h(\vec{x}_s), & \vec{x}_s \in \text{boundary} \end{cases}$$

then from (10).

$$u(\vec{x}) = u(r, \theta) = \frac{a}{2\pi} \int_0^{2\pi} h(\theta_0) \frac{(a^2 - r^2)/a^2}{r^2 + a^2 - 2ra \cos \phi} a d\theta_0$$

$$\text{or } \boxed{u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} h(\theta_0) \frac{a^2 - r^2}{r^2 + a^2 - 2ra \cos \phi} d\theta_0}$$

Now,



To obtain (9.5.61) in book.

HWK problems due Next Wed: 10 Haberman 9.5.5, 9.5.7, 9.5.10,
9.5.11, 9.5.14, 9.5.16.