

Fundamental Solutions for Poisson's eqn or Infinite space Green's function.

Consider Poisson's equation in the infinite space $\mathbb{R}^3$ or the infinite plane $\mathbb{R}^2$.

$$\begin{cases} \nabla^2 u(\vec{x}) = f(\vec{x}), & \vec{x} \in \mathbb{R}^2 \\ + \text{B.C. at } \infty \text{ (to be determined).} \\ r \rightarrow \infty \end{cases}$$

In polar coordinates:

$$\begin{cases} \nabla_{r,\theta}^2 u(r,\theta) = \frac{1}{r} \frac{d}{dr} \left(r \frac{du}{dr} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = f(r,\theta) \\ + \text{B.C. when } r \rightarrow \infty. \end{cases} \quad r \equiv \sqrt{x^2 + y^2}.$$

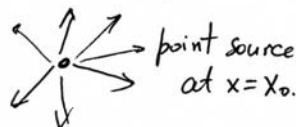
The corresponding BVP for the Green's fun. for a source located at $\vec{x} = \vec{x}_0$, and $\vec{r} \equiv \sqrt{(x-x_0)^2 + (y-y_0)^2}$.

$$\begin{cases} \nabla_{r,\theta}^2 G(\vec{x}, \vec{x}_0) = \delta(\vec{x} - \vec{x}_0), & \vec{x} \in \mathbb{R}^2 \\ + \text{B.C. (homog.) when } r \rightarrow \infty \end{cases}$$

Now, the response due to a point source

at $\vec{x} = \vec{x}_0$ is symmetric in the radial direction

$$\Rightarrow \frac{\partial}{\partial \theta} = 0.$$



Thus, the Green's fn satisfies:

$$\begin{cases} \frac{1}{r} \frac{d}{dr} \left(r \frac{dG}{dr} \right) (\vec{x}, \vec{x}_0) = \delta(\vec{x} - \vec{x}_0), \vec{x} \in \mathbb{R}^2 \\ + \text{B.C. at } r \rightarrow \infty. \end{cases}$$

If $\vec{x} \neq \vec{x}_0$

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dG}{dr} \right) = 0 \Rightarrow r \frac{dG}{dr} = C_1$$

$$\Rightarrow \frac{dG}{dr} = \frac{C_1}{r} \Rightarrow \boxed{G(\vec{x}, \vec{x}_0) = C_1 \ln(r) + C_2}, \vec{x} \neq \vec{x}_0. \quad (10.1)$$

The unknown constants can be obtained from an equivalent of the Jump Condition. This is

$$\iint_{S_\varepsilon(\vec{x}_0)} \nabla^2 G(\vec{x}, \vec{x}_0) dS_{\vec{x}} = \iint_{S_\varepsilon(\vec{x}_0)} \delta(\vec{x} - \vec{x}_0) dS_{\vec{x}} = 1$$

$\underbrace{S_\varepsilon(\vec{x}_0)}_{\text{Small circle of radius } \varepsilon}$

$$\stackrel{\text{div thm}}{\Rightarrow} \int_{C_\varepsilon} \underbrace{\nabla G(\vec{x}, \vec{x}_0) \cdot \hat{n}(\vec{x})}_{\frac{\partial G}{\partial r}(\vec{x}, \vec{x}_0)} d\ell_{\vec{x}} = 1$$

$$\Rightarrow 2\pi\varepsilon \left. \frac{\partial G}{\partial r} \right|_{r=\varepsilon} = 1 \Rightarrow \left. \frac{\partial G}{\partial r} \right|_{r=\varepsilon} = \frac{1}{2\pi\varepsilon}$$

$$\text{From (10.1)} \quad \left. \frac{\partial G}{\partial r} \right|_{r=\varepsilon} = \frac{C_1}{\varepsilon} \Rightarrow C_1 = \frac{1}{2\pi}.$$

$$\Rightarrow G(\vec{x}, \vec{x}_0) = \frac{1}{2\pi} \ln(r) + C_2.$$

unit except a constant C_2 . For convenience, $C_2 = 0$.

$$\Rightarrow \boxed{G(\vec{x}, \vec{x}_0) = \frac{1}{2\pi} \ln(r) \text{ in } \mathbb{R}^2}$$

Now, using Green's formula: over a large circle of radius r .

$$\iint_{\mathbb{R}^2} [u \nabla^2 G - G \nabla^2 u] dA_{\vec{x}} = \oint_{C_r} (u \nabla G - G \nabla u) \cdot \hat{n} d\ell_{\vec{x}}$$

We will impose the condition that 0 when $r \rightarrow \infty$.

$$\Rightarrow \boxed{u(\vec{x}) = \iint_{\mathbb{R}^2} G(\vec{x}, \vec{x}_0) f(\vec{x}_0) dS_{\vec{x}_0}}$$

(reversing roles
 $\vec{x} \leftrightarrow \vec{x}_0$
and using symmetry
 $G(\vec{x}, \vec{x}_0) = G(\vec{x}_0, \vec{x})$)

Rmk: The additional condition

$$\lim_{r \rightarrow \infty} \oint_{C_r} (u \nabla G - G \nabla u) \cdot \hat{n} d\ell_{\vec{x}} = 0 \text{ is equivalent to}$$

$$\lim_{r \rightarrow \infty} \int_{r \text{ constant}} u \frac{\partial G}{\partial r} - G \frac{\partial u}{\partial r} (r d\theta) = 2\pi r \left[u \frac{\partial G}{\partial r} - G \frac{\partial u}{\partial r} \right] \xrightarrow{r \rightarrow \infty} 0$$



or simply

$$\lim_{r \rightarrow \infty} r \left(u \frac{\partial G}{\partial r} - G \frac{\partial u}{\partial r} \right) = \lim_{r \rightarrow \infty} r \left[u \frac{1}{2\pi r} - \frac{1}{2\pi} \ln(r) \frac{\partial u}{\partial r} \right] = 0$$

$$\Rightarrow \lim_{r \rightarrow \infty} \left[u - r \ln(r) \frac{\partial u}{\partial r} \right] = 0.$$

This condition is satisfied for example if $u \sim \frac{1}{r}$ when $r \rightarrow \infty$.