Incompressible and Inviscid Fluid Equations

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1 Introduction

In this file, we will develop the fluids equation for conservation of mass or *Continuity Equation* and special form of the conservation of momentumn or *Euler Equation*.

2 The Continuity Equation

In this section we will derive the continuity equation. For this derivation we will be working with some volume of water, which we will call D. This could be a lake, the ocean, or some more controlled system such as a tank. Later when we begin our work with waves, we will specify exactly what we want D to be.

Let $\mathbf{u} = (u_1, u_2, u_3)$ be a velocity field, where $u_i = u_i(x, y, z, t)$ for $i = 1, 2, 3$.

The density of the water at a point (x, y, z) and at time t will be denoted as $\rho(x, y, z, t)$.

Let V be some arbitrary control bounded volume contained in D, and S the piecewise smooth and closed surface of V.

The mass of V can be written as $m = \int_V \rho(x, y, z, t)\, dV$.

The rate of change in mass for the volume V can then be written as $d/dt \left(\int_V \rho\, dV\right)$.

The principle of conservation of mass established that the rate of change of mass inside V equals the net mass that passes through the surface S per unit of time. Then,

$$\int_V \frac{\partial \rho}{\partial t} = -\int_S \rho \mathbf{u} \cdot \mathbf{n}\, dS, \tag{2.1}$$

where $\mathbf{n}$ is the outward unit normal. A quick check shows that the units work out correctly. In fact, $\frac{kg}{s} = \frac{(kg)(m^3)}{(m^3)(s)} = \frac{(kg)(m)(m^2)}{(m^3)(s)}$.

The integral on the left is obtained by applying Leibniz's rule. It means

$$\frac{d}{dt}\int_V \rho\, dV = \int_V \frac{\partial \rho}{\partial t}\, dV$$

The conditions are that
$\rho(\mathbf{x}, t)$ is an integrable function on V for each $t \in \mathbb{R}^+$ and
$\frac{\partial \rho}{\partial t}(\mathbf{x}, t)$ exists and is continuous in $V \times \mathbb{R}^+$.

Remark 2.1. The negative in the equation (2.1) can be explained as follows: Adding mass to a system results in a positive change of mass per unit of time. This flow of mass occurs through the surface S in the direction of the normal. Since S points outward, it must be negated to reflect a positive inflow of mass.

The Divergence Theorem applied on (2.1) leads to

$$\begin{aligned} &\int_V \frac{\partial \rho}{\partial t}\, dV = -\int_S (\rho \mathbf{u}) \cdot \mathbf{n}\, dS = -\int_V \nabla \cdot (\rho \mathbf{u})\, dV \\ &\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u})\right] dV = 0. \end{aligned} \tag{2.2}$$

Since the volume V is arbitrary, then the integrand should be zero, *i. e.,*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{2.3}$$

which is known as the continuity equation.

Question 1. If the fluid is incompressible, then the density is constant. Show that the continuity equation reduces to

$$\nabla \cdot \mathbf{u} = 0. \tag{2.4}$$

Assuming constant density for water is a common assumption, which we will make for the remainder of this paper.

3 Euler's Equation of Motion

Now, we will derive Euler's equation of motion. Now, we will use the principle of *Conservation of Momentum*. We are still considering the same domain D, velocity field $\mathbf{u}$, and volume V with a surface S.

Previous to the derivation, we will proof some lemmas that will be used later.

Question 2. Prove the following three lemmas:

Lemma 3.1. *If $f(x, y, z)$ is a scalar differentiable function in V and $\mathbf{u}(x, y, z)$ is a differentiable vector function in V, then*

$$\nabla \cdot (f\mathbf{u}) = \nabla f \cdot \mathbf{u} + f\nabla \cdot (\mathbf{u})$$

Prove this lemma introducing the components of $f\mathbf{u}$.

Lemma 3.2. *If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is a differentiable vector function, then*

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \langle \mathbf{u} \cdot \nabla u_1, \mathbf{u} \cdot \nabla u_2, \mathbf{u} \cdot \nabla u_3 \rangle$$

Lemma 3.3. *If f is a continuously differentiable scalar function, V a bounded region in the space, and S the surface enclosing the region V, then*

$$\int_S f\mathbf{n}\, dS = \int_V \nabla f\, dV$$

where $\mathbf{n}$ is the outward unit normal.

To derive Euler's equation, we will consider the law of conservation of momentum.
The momentum, I inside the volume V, can be written as $I = \int_V (\rho \mathbf{u})\, dV$,
then the rate of change of momentum is given by $dI/dt = d/dt \int_V (\rho \mathbf{u})\, dV$.
Also the net inflow of momentum in V through the surface S is given by $-\int_S (\rho \mathbf{u})\mathbf{u} \cdot \mathbf{n}\, dS$.
By the Law of Conservation of Momentum, the rate of change of momentum within the volume V equals the net inflow of momentum through the surface S plus the force $\mathbf{F}_p$ due to the pressure forces acting on the surface S and the force $\mathbf{F}_e$ due to the body forces per unit of mass acting inside the volume V. Thus we have

$$\frac{dI}{dt} = \frac{d}{dt}\int_V (\rho \mathbf{u})\, dV = -\int_S (\rho \mathbf{u})\mathbf{u} \cdot \mathbf{n}\, dS + \mathbf{F}, \tag{3.1}$$

where $\mathbf{F} = \mathbf{F}_p + \mathbf{F}_e$ A quick check insures that the units work out correctly, $\frac{(kg)(m)}{(s^2)} = \frac{(kg)(m)(m^3)}{(s)(m^3)(s)} = \frac{(kg)(m)(m)(m^2)}{(m^3)(s)(s)}$.

Upon expanding (3.1) component wise, we get

$$\int_V \frac{\partial}{\partial t}(\rho \mathbf{u})\, dV = -\int_S \langle (\rho u_1 \mathbf{u}) \cdot \mathbf{n}, (\rho u_2 \mathbf{u}) \cdot \mathbf{n}, (\rho u_3 \mathbf{u}) \cdot \mathbf{n} \rangle\; dV + \mathbf{F}.$$

Question 3. Using Green's theorem, the incompressibility condition and the above lemmas show that

$$\int_V \rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right] dV = \mathbf{F}. \tag{3.2}$$

Notice that $\mathbf{F}$ is the total force acting on the fluid, so we can break it up so that $\mathbf{F} = \mathbf{F}_p + \mathbf{F}_e$, where $\mathbf{F}_p$ is the force due to pressure, and $\mathbf{F}_e$ is some other external force. Pressure is force per unit area, so we can write

$$\mathbf{F}_p = -\int_S P\mathbf{n}\, dS \tag{3.3}$$

where $P = P(x, y, z, t)$ is the pressure. Again note the negative sign since the unit normal is outward. Also, if $\mathbf{A}_e$ represents the body forces per unit of mass, then the external force $\mathbf{F}_e$ is given by

$$\mathbf{F}_e = \int_V \rho \mathbf{A}_e\, dV \tag{3.4}$$

By substituting (3.3), and (3.4) into (3.2) and applying the Lemma 3.3 arrive to the final Euler's equation of motion given by

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{A}_e - \frac{\nabla P}{\rho}. \tag{3.5}$$