# Existence and Uniqueness of Solutions for BVP for Poisson's equation.

## Fredholm Alternative.

Consider the 2-D or 3-D Poisson equation

$$\boxed{\nabla^2 u(\vec{x}) = f(\vec{x}),} \quad \vec{x} \in \Omega \text{ (bounded region)} \tag{1}$$

with homogeneous BC's:

$$\boxed{\alpha u(\vec{x}_s) + \beta \nabla u(\vec{x}_s) \cdot \hat{n}(\vec{x}_s) = 0,} \quad \vec{x}_s \in \partial\Omega \text{ piece-wise smooth boundary.} \tag{2}$$

where $f(\vec{x})$ is a piecewise smooth function.

We would like to determine conditions for

- Existence of solutions for BVP (1)-(2).
- Establish uniqueness.

In order to do this, we will study the associated eigenvalue problem:

$$\begin{cases} \nabla^2 u(\vec{x}) = -\lambda u(\vec{x}), & \vec{x} \in \Omega. \quad (3) \\ \alpha u(\vec{x}_s) + \beta \nabla u(\vec{x}_s) \cdot \hat{n}(\vec{x}_s) = 0, & \vec{x}_s \in \partial\Omega. \quad (4) \end{cases}$$

In Sections 7.4 and 7.5, it's established that this is a multidimensional Sturm-Liouville EVP and satisfies similar properties:

a) All eigenvalues are real
b) There are infinitely many. $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$
c) The eigenfncs. form a complete set in $L_2(\Omega)$.
d) Eigenfncs. corresponding to different eigenvalues are orthogonal.

subject to the boundary conditions that we have discussed (excluding periodic and singular cases):

$$\boxed{\begin{aligned} \beta_1\phi(a) + \beta_2\frac{d\phi}{dx}(a) &= 0 \\ \beta_3\phi(b) + \beta_4\frac{d\phi}{dx}(b) &= 0, \end{aligned}} \tag{5.3.3}$$

where $\beta_i$ are real. In addition, to be called regular, the coefficients $p, q$, and $\sigma$ must be real and continuous everywhere (including the end points) and $p > 0$ and $\sigma > 0$ everywhere (also including the endpoints). For the regular Sturm-Liouville eigenvalue problem, many important general theorems exist. In Sec. 5.5 we will prove these results, and in Secs. 5.7 and 5.8 we will develop some more interesting examples that illustrate the significance of the general theorems.

**Statement of theorems.** At first let us just state (in one place) all the theorems we will discuss more fully later (and in some cases prove). For any regular Sturm-Liouville problem, all of the following theorems are valid:

1. All the eigenvalues $\lambda$ are real.
2. There exist an infinite number of eigenvalues:
$$\lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots$$
   a. There is a smallest eigenvalue, usually denoted $\lambda_1$.
   b. There is not a largest eigenvalue and $\lambda_n \to \infty$ as $n \to \infty$.
3. Corresponding to each eigenvalue $\lambda_n$, there is an eigenfunction, denoted $\phi_n(x)$ (which is unique to within an arbitrary multiplicative constant). $\phi_n(x)$ has exactly $n-1$ zeros for $a < x < b$.
4. The eigenfunctions $\phi_n(x)$ form a "complete" set, meaning that any piecewise smooth function $f(x)$ can be represented by a generalized Fourier series of the eigenfunctions:
$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x).$$
   Furthermore, this infinite series converges to $[f(x+) + f(x-)]/2$ for $a < x < b$ (if the coefficients $a_n$ are properly chosen).
5. Eigenfunctions belonging to different eigenvalues are orthogonal relative to the weight function $\sigma(x)$. In other words,
$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx = 0 \quad \text{if } \lambda_n \neq \lambda_m.$$
6. Any eigenvalue can be related to its eigenfunction by the **Rayleigh quotient**:
$$\lambda = \frac{-p\phi\, d\phi/dx\big|_a^b + \int_a^b [p(d\phi/dx)^2 - q\phi^2]\,dx}{\int_a^b \phi^2\sigma\,dx},$$
   where the boundary conditions may somewhat simplify this expression.

where $p$, $q$ and $\sigma$ are functions of $x$, $y$, and $z$. This eigenvalue problem [with boundary condition (7.4.2)] is directly analogous to the one-dimensional regular Sturm-Liouville eigenvalue problem. We prefer to deal with a somewhat simpler case, (7.4.1), corresponding to $p = \sigma = 1$ and $q = 0$. We will state and prove results for (7.4.1). We leave the discussion of (7.4.3) to some exercises (in Sec. 7.5).

Only for very simple geometries (for example, rectangles, see Sec. 7.3, or circles, see Sec. 7.7) can (7.4.1) be solved explicitly. In other situations, we may have to rely on numerical treatments. However, certain general properties of (7.4.1) are quite useful, all analogous to results we understand for the one-dimensional Sturm-Liouville problem. The reasons for the analogy will be discussed in the next section. We begin by simply stating the theorems for the two-dimensional case of (7.4.1) and (7.4.2):

1. All the eigenvalues are real.
2. There exists an infinite number of eigenvalues. There is a smallest eigenvalue, but no largest one.
3. Corresponding to an eigenvalue, there *may* be many eigenfunctions (unlike regular Sturm-Liouville eigenvalue problems).
4. The eigenfunctions $\phi(x, y)$ form a "complete" set, meaning that any piecewise smooth function $f(x, y)$ can be represented by a generalized Fourier series of the eigenfunctions:

$$f(x, y) \sim \sum_{\lambda} a_\lambda \phi_\lambda(x, y). \tag{7.4.4}$$

   Here $\sum_\lambda a_\lambda \phi_\lambda$ means a linear combination of all the eigenfunctions. The series converges in the mean if the coefficients $a_\lambda$ are chosen correctly.
5. Eigenfunctions belonging to different eigenvalues ($\lambda_1$ and $\lambda_2$) are orthogonal relative to the weight $\sigma(\sigma = 1)$ over the entire region $R$. Mathematically,

$$\iint_R \phi_{\lambda_1} \phi_{\lambda_2} \, dx \, dy = 0 \quad \text{if } \lambda_1 \neq \lambda_2, \tag{7.4.5}$$

   where $\iint_R dx \, dy$ represents an integral over the entire region $R$. Furthermore, different eigenfunctions belonging to the same eigenvalue can be made orthogonal by the Gram-Schmidt process (see Sec. 7.5). Thus, we may assume that (7.4.5) is valid even if $\lambda_1 = \lambda_2$ as long as $\phi_{\lambda_1}$ is independent of $\phi_{\lambda_2}$.
6. An eigenvalue $\lambda$ can be related to the eigenfunction by the Rayleigh quotient:

$$\lambda = \frac{-\oint \phi \nabla \phi \cdot \hat{\boldsymbol{n}} \, ds + \iint_R |\nabla \phi|^2 \, dx \, dy}{\iint_R \phi^2 \, dx \, dy}. \tag{7.4.6}$$

   The boundary conditions often simplify the boundary integral.

Construction of the Solution(s) for (1)-(2). (Existence)

Using the set of eigenfunctions, the solution for (1)-(2) (if it exists) can be constructed.

In fact, expanding both

$$u(\vec{x}) = \sum_{\lambda} a_\lambda \phi_\lambda(\vec{x})$$

Triple (3-D) or Double (2D) Series.

$$f(\vec{x}) = \sum_{\lambda} f_\lambda \phi_\lambda(\vec{x})$$

in terms of the eigenfunctions of the associated EVP and substituting them into equation (1), we obtain

$$\nabla^2\left(\sum_{\lambda} a_\lambda \phi_\lambda\right) = \sum_{\lambda} f_\lambda \phi_\lambda$$

$$\Rightarrow \sum_{\lambda} a_\lambda(\nabla^2\phi_\lambda) = \sum_{\lambda} f_\lambda \phi_\lambda$$

$$\Rightarrow \sum_{\lambda} a_\lambda(-\lambda\phi_\lambda) = \sum_{\lambda} f_\lambda \phi_\lambda$$

$$\Rightarrow -\lambda a_\lambda = f_\lambda, \quad \text{for all } \lambda.$$

If all eigenvalues $\lambda \neq 0$, then

$$a_\lambda = -\frac{f_\lambda}{\lambda}, \quad \text{for all } \lambda\text{'s} \qquad (2.1)$$

and a solution $u(\vec{x})$ of equation (1) can be defined as

$$u(\vec{x}) = \sum_{\lambda} a_\lambda \phi_\lambda \qquad (2.2)$$

where $a_\lambda$ is given by (2.1) and $f_\lambda \equiv \dfrac{\int_\Omega f(\vec{x}_0)\phi_\lambda(\vec{x}_0)\,dV_{\vec{x}_0}}{\int_\Omega \phi_\lambda^2(\vec{x}_0)\,dV_{\vec{x}_0}}$

$u(\vec{x})$ defined by (2.2) also satisfies the homogeneous BC's because every $\phi_\lambda$ satisfies it.

Therefore, $u(\vec{x})$ is a solution for the BVP (1)-(2).

This solution is also unique, if $\lambda = 0$ is not an eigenvalue, since the associated homogeneous BVP:

$$\begin{cases} \nabla^2 u(\vec{x}) = 0 \\ \alpha\, u(\vec{x}_s) + \beta\, \nabla u \cdot \hat{n}(\vec{x}_s) = 0 \end{cases}$$

only admits the trivial solution. Otherwise, $\lambda = 0$ would be an eigenvalue.

If $\lambda_0 = 0$ is an eigenvalue then there are two possibilities:

i) There are infinitely many solutions if

$$f_{\lambda_{0,i}} \equiv \frac{\int_\Omega f(\vec{x}_0)\, \phi_{\lambda_{0,i}}(\vec{x}_0)\, dV_{\vec{x}_0}}{\int_\Omega \phi^2_{\lambda_{0,i}}\, dV_{\vec{x}_0}} = 0$$

$\Leftrightarrow \quad f \perp \phi_{\lambda_{0,i}}$. $f$ is orthogonal to all linearly indep eigenfns. corresponding to $\lambda_0 = 0$.

In this case,

$$u(x) = C_1 \phi_{\lambda_{0,1}} + C_2 \phi_{\lambda_{0,2}} + \cdots + C_n \phi_{\lambda_{0,n}} + \sum_{\lambda \neq \lambda_{0,i}} a_\lambda \phi_\lambda$$

with $C_1, C_2, \ldots, C_n$ arbitrary real numbers.

ii) There are no solutions if

$f_{\lambda_{0;i}} \neq 0$, for at least one of the lin. indep corresponding eigenfn. $\phi_{\lambda_{0;i}}$

i.e., $$\int_{\Omega} f(\vec{x}_0)\, \phi_{\lambda_{0;i}}(\vec{x}_0)\, dV_{\vec{x}_0} \neq 0.$$

or $f(\vec{x}) \not\perp \phi_{\lambda_{0;i}}(\vec{x})$. $f$ is not orthogonal to $\phi_{\lambda_{0;i}}$

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## Representation of the solution and Green's function.

If $\lambda = 0$ is not an eigenvalue, then there is a unique solution given by

$$u(\vec{x}) = \sum_{\lambda}{}' a_\lambda \phi_\lambda(\vec{x}).$$

Replacing $a_\lambda = -\frac{f_\lambda}{\lambda}$,

$$u(\vec{x}) = \sum_{\lambda}{}' \left( \frac{\int_\Omega f(\vec{x}_0)\, \phi_\lambda(\vec{x}_0)\, dV_{\vec{x}_0}}{-\lambda \int_\Omega \phi_\lambda^2(\vec{x}_0)\, dV_{\vec{x}_0}} \right) \phi_\lambda(\vec{x})$$

Interchanging $\sum_\lambda \leftrightarrow \int_\Omega$.

$$u(\vec{x}) = \int_\Omega \Bigg( \overbrace{\sum_{\lambda}{}' \frac{\phi_\lambda(\vec{x}_0)\ \phi_\lambda(\vec{x})}{-\lambda \int_\Omega \phi_\lambda^2(\vec{x}_0)\, dV_{\vec{x}_0}}}^{\equiv\, G(\vec{x}, \vec{x}_0)} \Bigg) f(\vec{x}_0)\, dV_{\vec{x}_0}$$

The previous results can be summarized in the following theorem:

Thm.- (Fredholm Alternative).

1.- The BVP (1)-(2) for Poisson's equation has a unique solution if $\lambda = 0$ is not an eigenvalue of the associated EVP (3)-(4).

This unique solution can be represented by

$$u(x) = \int_\Omega \left[ \sum_\lambda \frac{\phi_\lambda(\vec{x}_0)\, \phi_\lambda(\vec{x})}{-\lambda \int_\Omega \phi_\lambda^2(\vec{x}_0)\, dv_{\vec{x}_0}} \right] f(x_0)\, dv_{\vec{x}_0} \qquad (5.1)$$

where $\{\phi_\lambda\}$ are the eigenfunctions.

2.- If $\lambda = 0$ is an eigenvalue of the associated EVP (3)-(4) with $\phi_{\lambda_{0,1}}, \ldots, \phi_{\lambda_{0,n}}$ linearly independent eigenfunctions

a) then BVP (1)-(2) has infinitely many solutions if

$$f \perp \phi_{\lambda_{0,i}}, \quad i = 1, \ldots n$$

and

$$u(x) = C_1 \phi_{\lambda_{0,1}} + \cdots + C_n \phi_{\lambda_{0,n}} + \sum_{\lambda \neq 0} a_\lambda \phi_\lambda$$

b) BVP (1)-(2) has no solutions if there is a $\phi_{\lambda_{0,j}}$ eigenfunction corresponding to $\lambda = 0$ such that $f(\vec{x})$ is not orthogonal to $\phi_{\lambda_{0,j}}$.

If $\lambda=0$ is not an eigenvalue the solution (5.1) can also be written as

$$u(x) = \int_{\Omega} G(\vec{x},\vec{x}_0)\, f(\vec{x}_0)\, d v_{\vec{x}_0}$$

where

$$G(\vec{x},\vec{x}_0) = \sum_{\lambda} \frac{\phi_\lambda(\vec{x}_0)\,\phi_\lambda(\vec{x})}{-\lambda \int_{\Omega} \phi_\lambda^2(\vec{x}_0)\, d v_{\vec{x}_0}} \qquad (6.1)$$

is the Green's function of the Laplace differential operator with the same homogeneous BC's.

It means $G(\vec{x},\vec{x}_0)$ satisfies the BVP:

$$\begin{cases} \nabla^2 G(\vec{x},\vec{x}_0) = \delta(\vec{x}-\vec{x}_0), & \vec{x} \in \Omega \quad (6.2) \\ \alpha\, G(\vec{x}_s,\vec{x}_0) + \beta \dfrac{\partial G}{\partial n}(\vec{x}_s,\vec{x}_0) = 0, & \vec{x}_s \in \partial\Omega \quad (6.3) \end{cases}$$

The expression for the Green's fn (6.1) can be obtained directly from (6.2) and (6.3) if we expand $G(\vec{x},\vec{x}_0)$ and $\delta(\vec{x}-\vec{x}_0)$ in terms of the eigenfunctions of the associated EVP (3)-(4). In fact, assuming that this is possible and also that the differentiation term by term is allowed, we obtain

$$G(\vec{x}, x_0) = \sum_{\lambda}' g_\lambda \phi_\lambda(\vec{x})$$

$$\delta(\vec{x}, \vec{x}_0) = \sum_{\lambda}' C_\lambda \phi_\lambda(x), \quad \text{where} \quad C_\lambda = \frac{\int_\Omega \phi_\lambda(\vec{x})\, \delta(\vec{x}, \vec{x}_0)\, dV_{\vec{x}}}{\int_\Omega \phi_\lambda^2(\vec{x}_0)\, dV_{\vec{x}_0}} = \frac{\phi_\lambda(\vec{x}_0)}{\int_\Omega \phi_\lambda^2(\vec{x}_0)\, dV_{\vec{x}_0}}$$

Subst. into equ.

$$\nabla^2 \left( \sum_\lambda g_\lambda \phi_\lambda(\vec{x}) \right) = \sum_\lambda \left( \frac{\phi_\lambda(\vec{x}_0)}{\int_\Omega \phi_\lambda^2(\vec{x}_0)\, dV_{\vec{x}_0}} \right) \phi_\lambda(\vec{x})$$

$$\Longleftrightarrow \quad \sum_\lambda g_\lambda \nabla^2(\phi_\lambda) = \sum_\lambda \left( \quad \right) \phi_\lambda(\vec{x}).$$

$$\Longleftrightarrow \quad \sum_\lambda' g_\lambda (-\lambda \phi_\lambda) = \sum_\lambda (\ ) \phi_\lambda.$$

$$\Longrightarrow \quad -\lambda g_\lambda = \frac{\phi_\lambda(\vec{x}_0)}{\int_\Omega \phi_\lambda^2(\vec{x}_0)\, dV_{\vec{x}_0}}$$

$$\Longrightarrow \quad \boxed{g_\lambda = \frac{\phi_\lambda(\vec{x}_0)}{-\lambda \int_\Omega \phi_\lambda^2(\vec{x}_0)\, dV_{\vec{x}_0}}}$$

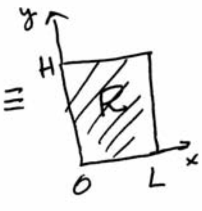
$$\Longrightarrow \quad \boxed{G(\vec{x}, \vec{x}_0) = \sum_\lambda \frac{\phi_\lambda(\vec{x}_0)\ \phi_\lambda(\vec{x})}{-\lambda \int_\Omega \phi_\lambda^2(\vec{x}_0)\, dV_{\vec{x}_0}}}$$

Rmk: All the assumptions made can be justified by a more rigorous definition of the $\delta$ function and Greens fn.

# Green's function for Poisson equation on a rectangular 2-D region with Dirichlet boundary conditions.

## Eigenfunction expansion:-

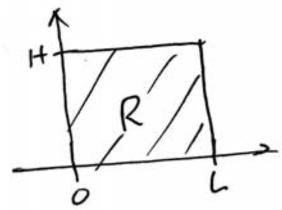
$$\begin{cases} \nabla^2 G(\vec{x};\vec{x}_0) = \delta(\vec{x}-\vec{x}_0), & \vec{x} \in \Omega \equiv \\ G(\vec{x}_s, \vec{x}_0) = 0, & \vec{x}_s \in \partial\Omega. \end{cases}$$




Dirichlet B.C. $\rightarrow$

## Related eigenvalue problem:

$$\begin{cases} \nabla^2\phi = \lambda\phi, & \phi(\vec{x}_s \\ \phi(\vec{x}_s) = 0, & \vec{x}_s \in \partial\Omega \end{cases}$$




Eigenvalues are $\lambda_{nm} = (n\pi/L)^2 + (m\pi/H)^2$, $n, m = 1, 2, 3, \ldots$

Corresponding eigenfns: $\phi_{nm} = \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right)$

and ~~Wher~~ $\iint_R \phi_{nm}^2(x,y)\,dx\,dy = \frac{L}{2}\cdot\frac{H}{2}.$

Using the formula obtained in our previous class:

$$G(\vec{x},\vec{x}_0) = \sum_\lambda \frac{\phi_\lambda(\vec{x})\,\phi_\lambda(\vec{x}_0)}{-\lambda\iint_{\partial\Omega}\phi_\lambda^2\,dA}$$

we obtain

$$G(\vec{x},\vec{x}_0) = \frac{-4}{LH}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right)\sin\left(\frac{n\pi}{L}x_0\right)\sin\left(\frac{m\pi}{H}y_0\right)}{\left(\frac{m\pi}{H}\right)^2+\left(\frac{n\pi}{L}\right)^2}$$

Definition of $\delta$ function.

Multidimensional Dirac delta function satisfies the same properties as in 1-dim.

a) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\vec{x})\,\delta(\vec{x}-\vec{x}_0)\,d\vec{x} = f(\vec{x}_0),$$

b) $$\delta(\vec{x}-\vec{x}_0) \equiv \begin{cases} 0, & \vec{x} \neq \vec{x}_0 \\ +\infty, & \vec{x} = \vec{x}_0 \end{cases}.$$

c) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\vec{x}-\vec{x}_0)\,dx = 1,$$

Green's formula for the Laplacian operator is also valid when one of the functions in it is a Green's fn.

$$\iiint_{\Omega} \left[u\nabla^2 v - v\nabla^2 u\right] dv = \iint_{\partial\Omega} \left[u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right] ds \quad (5)$$

Application: We can use Green's formula (5) to show that the Green's fn. is symmetric.

In fact, if

$u(\vec{x}) = G(\vec{x},\vec{x}_1)$ and $v(\vec{x}) = G(\vec{x},\vec{x}_2)$

$$\iiint_{\Omega} \left[G(\vec{x},\vec{x}_1)\,\delta(\vec{x}-\vec{x}_2) - G(\vec{x},\vec{x}_2)\,\delta(\vec{x}-\vec{x}_1)\right] dv = \overset{\text{Homog. BC's.}}{0}$$

$$\Rightarrow \iiint_{\Omega} G(\vec{x},\vec{x}_1)\, \delta(\vec{x}-\vec{x}_2)\, dv = \iiint_{\Omega} G(\vec{x},\vec{x}_2)\, \delta(\vec{x}-\vec{x}_1)\, dv$$

$$\Rightarrow \quad G(\vec{x}_2,\vec{x}_1) \quad = \quad G(\vec{x}_1,\vec{x}_2) \checkmark$$

It's Symmetric!

<u>Representation of the Solution of BVP (1) and (2)</u>
<u>When $\lambda = 0$ is not an eigenvalue. and $h(\vec{x}) \equiv 0$</u> $\to$ Homog. BC's.

Using Green's formula

$$\iiint_{\Omega} \left[ u(\vec{x})\, \nabla^2 G(\vec{x},\vec{x}_0) - G(\vec{x},\vec{x}_0)\, \nabla^2 u(\vec{x}) \right] dv =$$

$$\Rightarrow \iiint_{\Omega} u(\vec{x})\, \delta(\vec{x}-\vec{x}_0)\, dv = \iiint_{\Omega} G(\vec{x},\vec{x}_0)\, f(\vec{x})\, dv$$

$$\Rightarrow \quad u(\vec{x}_0) = \iiint_{G} G(\vec{x},\vec{x}_0)\, f(\vec{x})\, dv.$$

Reversing the roles of $\vec{x}$ and $\vec{x}_0$.

$$u(\vec{x}) = \iiint_{\Omega} G(\vec{x}_0,\vec{x})\, f(\vec{x}_0)\, dv$$

Using the Symmetry of $G(\vec{x},x_0)$.

$$\boxed{u(\vec{x}) = \iiint_{\Omega} G(\vec{x},\vec{x}_0)\, f(\vec{x}_0)\, dv.}$$

# Green's fns and nonhomog. BC's.

Consider
$$\begin{cases} \nabla^2 u = f(\vec{x}), & \vec{x} \in \Omega \\ u(\vec{x}_s) = h(\vec{x}_s), & \vec{x}_s \in \partial\Omega. \end{cases}$$

Associated Green's fn. Satisfies
$$\begin{cases} \nabla^2 G(\vec{x};\vec{x}_0) = \delta(\vec{x}-\vec{x}_0), & \vec{x} \in \Omega. \\ G(\vec{x}_s,\vec{x}_0) = 0, & \vec{x} \in \partial\Omega. \end{cases}$$

Green's formula:

$$\iiint_{\Omega} \left[u \nabla^2 G - G \nabla^2 u\right] dv_{\vec{x}} = \iint_{\partial\Omega} \left[u \nabla_{\vec{x}} G - \overset{0}{\cancel{G}} \nabla_{\vec{x}} u\right] ds_{\vec{x}}$$

$$\Rightarrow \iiint_{\Omega} \left[u(\vec{x})\delta(\vec{x}-\vec{x}_0) - G(\vec{x},\vec{x}_0) f(\vec{x})\right] dv_{\vec{x}} = \iint_{\partial\Omega} h(\vec{x}) \nabla_{\vec{x}} G(\vec{x},\vec{x}_0)\cdot\hat{n} \, ds_{\vec{x}}$$

$$\Rightarrow u(\vec{x}_0) = \iiint_{\Omega} G(\vec{x};\vec{x}_0) f(\vec{x}) \, dv_{\vec{x}} + \iint_{\partial\Omega} h(\vec{x}) \nabla_{\vec{x}} G(\vec{x},\vec{x}_0)\cdot\hat{n} \, ds_{\vec{x}}$$

Using Symmetry $u(\vec{x}_0) = \iiint_{\Omega} G(\vec{x}_0,\vec{x}) f(\vec{x}) \, dv_{\vec{x}} + \iint_{\partial\Omega} h(\vec{x}) \nabla_{\vec{x}} G(\vec{x}_0,\vec{x})\cdot\hat{n} \, ds_{\vec{x}}$

Interchanging the roles of $\vec{x}$ and $\vec{x}_0$.

$$\Downarrow$$

$$u(\vec{x}) = \iiint_{\Omega} G(\vec{x},\vec{x}_0) f(\vec{x}_0) \, dv_{\vec{x}_0} + \iint_{\partial\Omega} h(\vec{x}_0) \nabla_{\vec{x}_0} G(\vec{x},\vec{x}_0)\cdot\hat{n} \, ds_{\vec{x}_0}$$

Remark:

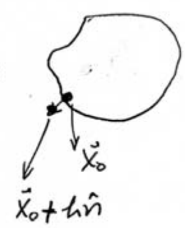
$$\nabla_{\vec{x}_0} G(\vec{x},\vec{x}_0)\cdot\hat{n} = \lim_{h\to 0} \frac{G(\vec{x},\vec{x}_0+h\hat{n}) - G(\vec{x},\vec{x}_0)}{h}$$

Interpretation:

1) $\dfrac{G(\vec{x},\vec{x}_0+h\hat{n})}{h}$ : response to positive source located at $\vec{x}=x_0+h\hat{n}$. of strength $\frac{1}{h}$.

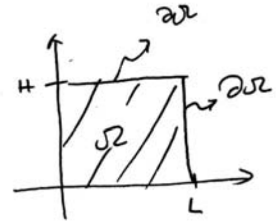
2) $-\dfrac{G(\vec{x},\vec{x}_0)}{h}$ : response to a negative source of strength $-\frac{1}{h}$ located at $\vec{x}=\vec{x}_0$

3) Influence function for nonhomog B.C is combination of the two sources described in (1) and (2). It's called dipole source. Nonhomog. B.C. equivalent to a surface distribution of dipoles.



## Direct solution of Green's function using 1-D eigenfunctions.

$$\begin{cases} \nabla^2 G(\vec{x}, \vec{x}_0) = \delta(\vec{x} - \vec{x}_0), \ \vec{x} \in \Omega & (4.1) \\ G(\vec{x}_s, \vec{x}_0) = 0, \quad \vec{x}_s \in \partial\Omega & (4.2) \end{cases}$$




Using one-dim. eigenfns. in $x$.

$$G(\vec{x}; \vec{x}_0) = \sum_{n=1}^{\infty} a_n(y) \sin \frac{n\pi}{L} x$$

Substitution in (4.1) leads to

$$\sum \left[ a_n''(y) - \left(\frac{n\pi}{L}\right)^2 a_n(y) \right] \sin \frac{n\pi}{L} x = \delta(x - x_0)\, \delta(y - y_0)$$

Using orthogonality,

$$a_n''(y) - \left(\frac{n\pi}{L}\right)^2 a_n(y) = \frac{1}{\int_0^L \sin^2 \frac{n\pi}{L} x \, dx} \int_0^L \delta(x - x_0)\, \delta(y - y_0) \sin \frac{n\pi}{L} x \, dx$$

$$= \frac{2}{L} \delta(y - y_0) \sin \frac{n\pi}{L} x_0 .$$

Also, $G(x, 0; x_0, y_0) = 0$, $\quad G(x, H; x_0, y_0) = 0$

$\Rightarrow \quad a_n(0) = 0$, and $a_n(H) = 0$

Therefore, we have

$$\begin{cases} a_n''(y) - \left(\frac{n\pi}{L}\right)^2 a_n(y) = \frac{2}{L}\sin\left(\frac{n\pi}{L}x_0\right)\delta(y-y_0) \\ a_n(0)=0, \qquad a_n(H)=0 \end{cases} \tag{5.1}$$

We have solved similar problems before
- obtain first general soln. of homogeneous equation. → 4 unkws. 2 each side.
- Use BC's to determine 2 of the 4 unkws.
- Use continuity of $a_n(y)$ and also jump in the derivative to obtain the remainder 2 unkws.

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<u>General Soln:</u> $a_n(y) = \bar{C}_{n_{\ell r}} \sinh\left(\frac{n\pi}{L}y\right) + \bar{D}_{n_{\ell r}} \cosh\left(\frac{n\pi}{L}y\right)$

B.C's ⇒
$$a_n(y) = \begin{cases} \bar{C}_{n\ell}\sinh\left(\frac{n\pi}{L}y\right), & y < y_0 \\ \bar{C}_{nr}\sinh\left(\frac{n\pi}{L}(y-H)\right), & y > y_0 \end{cases}$$

Continuity ⇒ $a_n(y_0^-) = a_n(y_0^+)$

$$\bar{C}_{n\ell}\sinh\left(\frac{n\pi}{L}y_0\right) = \bar{C}_{nr}\sinh\left(\frac{n\pi}{L}(y_0-H)\right)$$

Enough to ask for
$$\bar{C}_{n\ell} = \sinh\left(\frac{n\pi}{L}(y_0-H)\right)C_n$$
$$\bar{C}_{nr} = \sinh\left(\frac{n\pi}{L}y_0\right)C_n.$$

$$\Rightarrow \quad a_n(y) = \begin{cases} C_n \sinh\left(\frac{n\pi}{L}(y_0-H)\right) \sinh\left(\frac{n\pi}{L}y\right), & y < y_0 \\ C_n \sinh\left(\frac{n\pi}{L}y_0\right) \sinh\left(\frac{n\pi}{L}(y-H)\right), & y > y_0 \end{cases}$$

Jump in the derivative:

Integrating (5.1) $\int_{y_0-\varepsilon}^{y_0+\varepsilon}$ and taking $\lim_{\varepsilon \to 0}$

$$a_n'(y_0^+) - a_n'(y_0^-) = \frac{2}{L} \sin\left(\frac{n\pi}{L}x_0\right).$$

$$\Leftrightarrow C_n \frac{n\pi}{L}\left[\sinh\left(\frac{n\pi y_0}{L}\right)\cosh\left(\frac{n\pi(y_0-H)}{L}\right) - \sinh\left(\frac{n\pi(y_0-H)}{L}\right) * \right.$$
$$\left. * \cosh\left(\frac{n\pi}{L}y_0\right)\right] = \frac{2}{L}\sin\left(\frac{n\pi}{L}x_0\right).$$

$$\Leftrightarrow C_n \frac{n\pi}{\cancel{L}} * \sinh\left(\frac{n\pi}{L}\left(\cancel{y_0} - (\cancel{y_0} - H)\right)\right) = \frac{2}{\cancel{L}} \sin\left(\frac{n\pi}{L}x_0\right)$$

$$\Rightarrow \quad C_n = \frac{2}{n\pi} \frac{\sin\left(\frac{n\pi}{L}x_0\right)}{\sinh\left(\frac{n\pi H}{L}\right)}$$

Finally,

$$G(\vec{x};\vec{x}_0) = \sum_{n=1}^{\infty} \frac{2\sin\left(\frac{n\pi}{L}x_0\right)\sin\left(\frac{n\pi}{L}x\right)}{n\pi \sinh\left(\frac{n\pi}{L}H\right)} \begin{cases} \sinh\left(\frac{n\pi(y_0-H)}{L}\right)\sinh\frac{n\pi}{L}y, & y < y_0 \\ \sinh\left(\frac{n\pi(y-H)}{L}\right)\sinh\frac{n\pi}{L}y_0, & y > y_0. \end{cases}$$