

Newton - Raphson Method

(Exercise 11 Sect. 2.3)

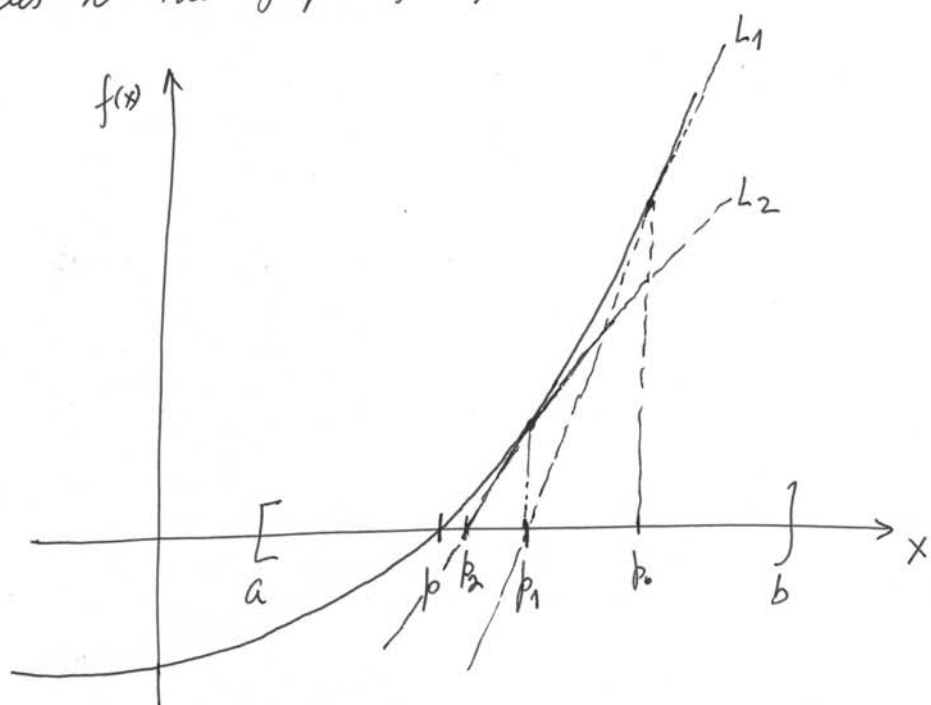
Assumptions: Given $y = f(x) \in C^2[a, b]$

$\downarrow$
 f, f' and f'' are
 Continuous in $[a, b]$.

and knowing $f(x) = 0$ has a root $p \in [a, b]$

and $f'(x) \neq 0$ on (a, b) .

Then, we can construct a sequence of tangent
 lines to the graph of $f(x)$ as follows:



Steps: $f(a) \cdot f(b) < 0$

1) Choose $p_0 \in (a, b)$

2) Construct tangent line through $(p_0, f(p_0))$: L_1

$$y - f(p_0) = f'(p_0)(x - p_0)$$

or $y = f(p_0) + f'(p_0)(x - p_0).$

3) Find intersection of L_1 with x -axis.

$$\underline{y=0} \quad (x^* - p_0) f'(p_0) = -f(p_0)$$

$$\Rightarrow x^* = p_0 - \frac{f(p_0)}{f'(p_0)} = p_1$$

4) Call $x^* = p_1$ and repeat process to construct L_2 through $(p_1, f(p_1))$ and find p_2 as the intersection of L_2 with the x -axis, i.e.,

$$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)}$$

After n steps

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

functional iteration.

Newton Iteration

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_n)} \quad (N.1)$$

It can be seen as a fixed point problem

$$g(x) = x - \frac{f(x)}{f'(x)} \quad (N.2)$$

Assuming $f'(x) \neq 0$.

It's clear from (N.2) the equivalence between the problems of finding a root p of $f(x)$ and finding a fixed point p of $g(x)$.

Theorem. - (Newton Method)

- ① $f \in C^2[a, b]$ (Twice continuously differentiable)
- ② there exists p in (a, b) , such that $f(p) = 0$.
- ③ $f'(p) \neq 0$.

Then, there exists $\delta > 0$, such that for $p_0 \in [p - \delta, p + \delta]$

Newton iteration (N.1) generates $\{p_n\}_{n=1}^{\infty}$ such that

$$p_n \xrightarrow{n \rightarrow \infty} p$$

Theorem:- (Fixed point)

- ① $g(x)$ is a Continuous function on $[a, b]$.
- ② The Values $g(x)$ belong to $[a, b]$. $g(x) \in [a, b]$
- ③ $g'(x)$ exists in (a, b)
- ④ there exists $0 < K < 1$, Such that

$$|g'(x)| \leq K, \quad x \in (a, b)$$

Then

- ① there is a unique fixed point $p \in [a, b]$.
- ② the Sequence $\{p_n\}_{n=1}^{\infty}$ obtained from the functional iteration

$$p_n = g(p_{n-1}), \quad n \geq 1,$$

where p_0 an arbitrary point in $[a, b]$,
Converges to the unique point p in $[a, b]$.

Proof. -

Since $f'(p) \neq 0$ and $f'(x)$ is cont in $[a, b]$
then, there exists $\delta_1 > 0$ such that $f'(x) \neq 0$
in $[p - \delta_1, p + \delta_1]$

Therefore,

$$g(x) = x - \frac{f(x)}{f'(x)}$$

is cont in $[p - \delta_1, p + \delta_1]$

because x , $f(x)$ and $f'(x)$ are all conts. in $[p - \delta_1, p + \delta_1]$

and $f'(x) \neq 0$ in $[p - \delta_1, p + \delta_1]$.

Next, we will find an interval $[p - \delta, p + \delta] \subset [p - \delta_1, p + \delta_1]$

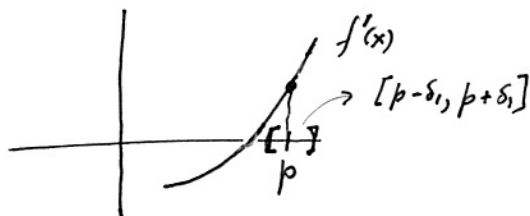
Such that

$$a) g(x) \in C[p - \delta, p + \delta], \quad b) g([p - \delta, p + \delta]) \subset [p - \delta, p + \delta]$$

$$c) |g'(x)| \leq K < 1$$

Once this conditions have been established, application
of the fixed point theorem to $g(x)$ leads to the
conclusion that the functional iteration of $g(x)$

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \xrightarrow{n \rightarrow \infty} p.$$



Condition (a)

To establish condition (a) is trivial, since

if $g(x) \in [p-\delta_1, p+\delta_1]$, then $g(x) \in C[p-\delta, p+\delta]$

for any $\delta > 0$ such that $[p-\delta, p+\delta] \subset [p-\delta_1, p+\delta_1]$.

Condition (c)

Applying quotient rule to $g(x)$

$$g'(x) = 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2}$$

This formula for $g'(x)$ is valid in $[p-\delta_1, p+\delta_1]$

because $f''(x)$ exists there and $f'(x) \neq 0$.

Thus $g'(x)$ exists in $[p-\delta_1, p+\delta_1]$.

Moreover, $\boxed{g'(x) \text{ is conts. in } [p-\delta_1, p+\delta_1]}$ because $(*)$

f, f' , and f'' are conts and $f'(x) \neq 0$ in $[p-\delta_1, p+\delta_1]$.

Furthermore,

$$\boxed{g'(p) = \frac{f(p)f''(p)}{[f'(p)]^2} = 0,} \quad \begin{matrix} \text{(by Hypothesis)} \\ f'(p) \neq 0 \\ (**) \end{matrix}$$

From (*) and (**), we conclude that there exists

an interval $[p-\delta, p+\delta] \subset [p-\delta_1, p+\delta_1]$



where

$$\boxed{|g'(x)| \leq K < 1}$$

Condition (c)

Condition (b)

Finally, using that $g(x) \in C[p-\delta, p+\delta]$ (cond. (a))
and that $g'(x)$ exists in $(p-\delta, p+\delta)$

For any $x \in [p-\delta, p+\delta]$:

$$|g(x) - p| = |g(x) - g(p)| \stackrel{\text{MVT}}{=} g'(\xi) |x - p| \leq K |x - p|$$

$\xi \in (p-\delta, p+\delta)$

And $K |x - p| < |x - p| < \delta$

Therefore $|g(x) - p| < \delta$, which is equivalent to

$$g(x) \in (p-\delta, p+\delta) \subset [p-\delta, p+\delta] \quad \left(\begin{array}{c} \text{Condition} \\ (b) \end{array} \right)$$

Since all hypothesis of fixed point theorem (a), (b), and (c) are satisfied, then the functional iteration

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \xrightarrow{n \rightarrow \infty} p.$$