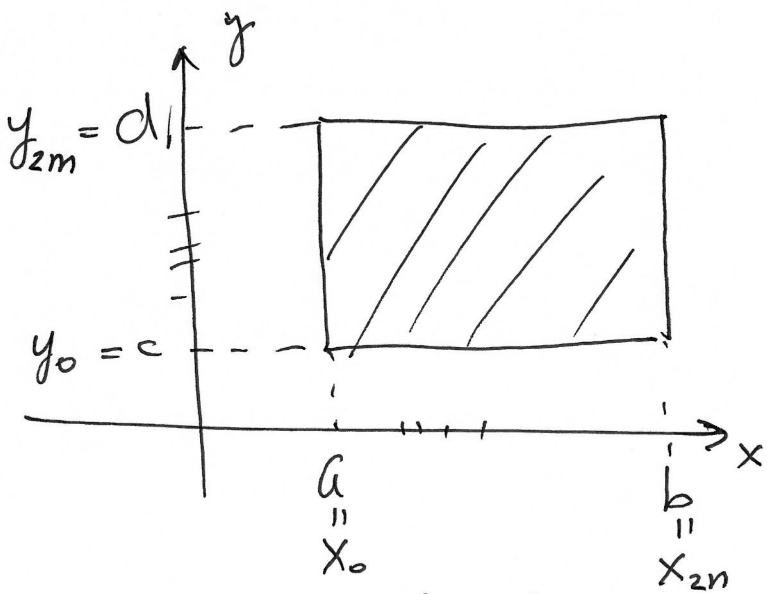
Composite Simpson's rule in 2D

$$\int_a^b \int_c^d f(x,y)\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$



Applying Comp. Simpson in "y".

$$= \int_a^b \left\{ \frac{k}{3} \left[f(x,y_0) + 2 \sum_{j=1}^{m-1}{}' f(x, y_{2j}) + 4 \sum_{j=1}^{m}{}' f(x, y_{2j-1}) + f(x, y_{2m}) \right] - \frac{d-c}{180} \frac{\partial^4 f}{\partial y^4}(x,\mu) \right\} dx$$

$$h = \frac{b-a}{2n}, \quad k = \frac{d-c}{2m}$$

$$= \frac{k}{3} \left[\underbrace{\int_a^b f(x,y_0)\,dx}_{= I_0} + 2 \sum_{j=1}^{m-1}{}' \underbrace{\int_a^b f(x, y_{2j})}_{= I_{2j}} + 4 \sum_{j=1}^{m} \underbrace{\int_a^b f(x, y_{2j-1})\,dx}_{= I_{2j-1}} + \underbrace{\int_a^b f(x, y_{2m})\,dx}_{= I_{2m}} \right]$$

$$\underbrace{- \frac{d-c}{180} \int_a^b \frac{\partial^4 f}{\partial y^4}(x,\mu)\,dx}_{= E_y}$$

Each Single Term transforms after integration:

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(I)
$$\int_a^b f(x,y_0)\,dx = \frac{h}{3}\left\{ f(x_0,y_0) + 2\sum_{i=1}^{m-1} f(x_{2i},y_0) + 4\sum_{i=1}^{n} f(x_{2i-1},y_0) + f(x_{2n},y_0)\right\} +$$
$$- \frac{b-a}{180} h^4 \frac{\partial^4 f}{\partial x^4}(\xi_0, y_0) = I_0 + E_0$$

(II)
$$2\sum_{j=1}^{m-1}\int_a^b f(x,y_{2j})\,dx = \frac{2h}{3}\left\{ \sum_{j=1}^{m-1} f(x_0,y_{2j}) + 2\sum_{j=1}^{m-1}\sum_{i=1}^{n-1} f(x_{2i},y_{2j}) + 4\sum_{j=1}^{m-1}\sum_{i=1}^{n} f(x_{2i-1},y_{2j}) + \right.$$
$$\left. + \sum_{j=1}^{m-1} f(x_{2n},y_{2j})\right\} + 2\sum_{j=1}^{m-1} (-1)\frac{b-a}{180} h^4 \frac{\partial^4 f}{\partial x^4}(\xi_{2j},y_{2j}) = I_{2j} + E_{2j}$$

(III)
$$4\sum_{j=1}^{m}\int_a^b f(x,y_{2j-1})\,dx = \frac{4h}{3}\left\{ \sum_{j=1}^{m} f(x_0,y_{2j-1}) + 2\sum_{j=1}^{m}\sum_{i=1}^{n-1} f(x_{2i},y_{2j-1}) + 4\sum_{j=1}^{m}\sum_{i=1}^{n} f(x_{2i-1},y_{2j-1}) + \right.$$
$$\left. + \sum_{j=1}^{m} f(x_{2n},y_{2j-1})\right\} + 4\sum_{j=1}^{m} (-1)\frac{b-a}{180} h^4 \frac{\partial^4 f}{\partial x^4}(\xi_{2j-1},y_{2j-1}) =$$
$$= I_{2j-1} + E_{2j-1}$$

(IV) $$\int_a^b f(x, y_{2m})\,dx = \frac{h}{3}\left\{ f(x_0, y_{2m}) + 2\sum_{i=1}^{n-1} f(x_{2i}, y_{2m}) + 4\sum_{i=1}^{n} f(x_{2i-1}, y_{2m}) + \right.$$
$$\left. + f(x_{2n}, y_{2m}) \right\} - \frac{b-a}{180} h^4 \frac{\partial^4 f}{\partial x^4}(\xi_{2m}, y_{2m}) = I_{2m} + E_{2m}$$

Substitution on (1) leads to

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \frac{K}{3}\left[I_0 + I_{2j} + I_{2j-1} + I_{2m} \right] + \underbrace{\frac{K}{3}\left[E_0 + E_{2j} + E_{2j-1} + E_{2m} \right] - \frac{(d-c)K^4}{180}\int_a^b \frac{\partial^4 f}{\partial y^4}(x, \mu)\,dx}_{= E_y} \qquad (2)$$

Assuming $\frac{\partial^4 f}{\partial x^4}(x,y)$ is Cont. on R.

there exist $\min\limits_{(x,y)\in R} \frac{\partial^4 f}{\partial x^4}(x,y)$ and $\max\limits_{(x,y)\in R} \frac{\partial^4 f}{\partial x^4}(x,y)$

On the other hand,

$$\frac{K}{3}\left[E_0 + E_{2j} + E_{2j-1} + E_{2m}\right] = -\frac{K}{3}\,\frac{(b-a)}{180}\,h^4\Bigg[\overbrace{\frac{\partial^4 f}{\partial x^4}(\xi_0, y_0)}^{=E_{4,0}} +$$

$$+\ 2\overbrace{\sum_{j=1}^{m-1}\frac{\partial^4 f}{\partial x^4}(\xi_{2j}, y_{2j-1})}^{=E_{4,2j}} + 4\overbrace{\sum_{j=1}^{m}\frac{\partial^4 f}{\partial x^4}(\xi_{2j-1}, y_{2j-1})}^{=E_{4,2j-1}} + \overbrace{\frac{\partial^4 f}{\partial x^4}(\xi_{2m}, y_{2m})}^{=E_{4,2m}}\Bigg]$$

$$= -\frac{K}{3}\,\frac{(b-a)}{180}\,h^4\left[E_{4,0} + 2E_{4,2j} + 4E_{4,2j-1} + E_{4,2m}\right] = -\frac{K}{3}\,\frac{(b-a)}{180}\,h^4\,\tilde{I}_4$$

Now,

$$(1+2(m-1)+4m+1)\min_R \frac{\partial^4 f}{\partial x^4} \le \tilde{I}_4 \le (1+2(m-1)+4m+1)\max_R \frac{\partial^4 f}{\partial x^4}$$

then

$$\min_R \frac{\partial^4 f}{\partial x^4} \le \frac{1}{6m}\tilde{I}_4 \le \max_R \frac{\partial^4 f}{\partial x^4}$$

$\therefore$ Intermediate Value thm $\Rightarrow$ $\frac{1}{6m}\tilde{I}_4 = \frac{\partial^4 f}{\partial x^4}(\hat{\eta}, \hat{\mu})$, $\quad(\hat{\eta}, \hat{\mu})$ in R.

Since, $k = \frac{d-c}{2m} \Rightarrow 6m = 3\,\frac{d-c}{k}$

$$\therefore \quad \frac{k}{3}\left[E_0 + E_{2j} + E_{2j-1} + E_{2m}\right] = -\frac{\cancel{k}}{\cancel{3}}\,\frac{(b-a)}{180}\,h^4\,\frac{\cancel{3}(d-c)}{\cancel{k}}\,\frac{\partial^4 f}{\partial x^4}(\hat{\eta}, \hat{\mu})$$

$$= -\frac{(b-a)(d-c)}{180}\,h^4\,\frac{\partial^4 f}{\partial x^4}(\hat{\eta}, \hat{\mu}).$$

Also, by using the Mean Value thm for integrals

$$E_y = -\frac{(d-c)k^4}{180}\int_a^b \frac{\partial^4 f}{\partial y^4}(x, \mu)\,dx = -\frac{(d-c)(b-a)}{180}\,k^4\,\frac{\partial^4 f}{\partial y^4}(\eta, \mu)$$

Finally,

$$\boxed{\int_a^b\int_c^d f(x,y)\,dy\,dx = \frac{k}{3}\left[I_0 + I_{2j} + I_{2j-1} + I_{2m}\right] - \frac{(b-a)(d-c)}{180}\left[h^4\,\frac{\partial^4 f}{\partial x^4}(\hat{\eta}, \hat{\mu}) + k^4\,\frac{\partial^4 f}{\partial y^4}(\eta, \mu)\right]}$$

Composite Simpson's rule in 2D.

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \frac{k}{3}\left[I_0 + I_{2j} + I_{2j-1} + I_{2m}\right] - \frac{(b-a)(d-c)}{180}\left[h^4 \frac{\partial^4 f}{\partial x^4}(\hat{\eta},\hat{\mu}) + k^4 \frac{\partial^4 f}{\partial y^4}(\eta,\mu)\right]$$

$a < \hat{\eta}, \eta < b$

$c < \mu, \hat{\mu} < d$

where

$$I_0 = \frac{h}{3}\left[f(x_0,y_0) + 2\sum_{i=1}^{n-1} f(x_{2i},y_0) + 4\sum_{i=1}^{n} f(x_{2i-1},y_0) + f(x_{2n},y_0)\right]$$

$$I_{2j} = \frac{2h}{3}\left[\sum_{j=1}^{m-1} f(x_0,y_{2j}) + 2\sum_{j=1}^{m-1}\sum_{i=1}^{n-1} f(x_{2i},y_{2j}) + 4\sum_{j=1}^{m-1}\sum_{i=1}^{n} f(x_{2i-1},y_{2j}) + \sum_{j=1}^{m-1} f(x_{2n},y_{2j})\right]$$

$$I_{2j-1} = \frac{4h}{3}\left[\sum_{j=1}^{m} f(x_0,y_{2j-1}) + 2\sum_{j=1}^{m}\sum_{i=1}^{n-1} f(x_{2i},y_{2j-1}) + 4\sum_{j=1}^{m}\sum_{i=1}^{n} f(x_{2i-1},y_{2j-1}) + \sum_{j=1}^{m} f(x_{2n},y_{2j-1})\right] \quad (6.1)$$

$$I_{2m} = \frac{h}{3}\left[f(x_0,y_{2m}) + 2\sum_{i=1}^{n-1} f(x_{2i},y_{2m}) + 4\sum_{i=1}^{n} f(x_{2i-1},y_{2m}) + f(x_{2n},y_{2m})\right]$$

Exercise Applying (6.1) to $\displaystyle\int_{1.4}^{2}\int_{1.0}^{1.5} \ln(x+2y)\,dy\,dx$

Using (6.1)

$j=0$ $\quad I_0 = \dfrac{0.15}{3}\Big[f(x_0,y_0)+4f(x_1,y_0) + 2f(x_2,y_0)+4f(x_3,y_0) + f(x_4,y_0)\Big]$

$n=2$

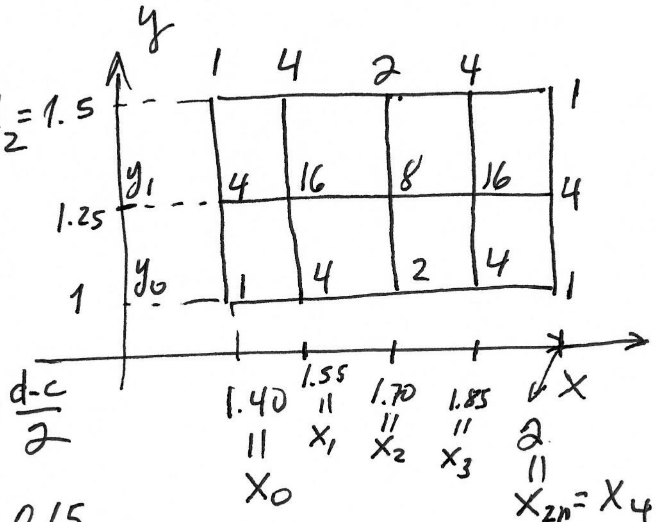
$\Downarrow$

$h=\dfrac{b-a}{4}$

$m=1 \Rightarrow k=\dfrac{d-c}{2}$

or $h=\dfrac{0.6}{4}=0.15$

$k=\dfrac{0.5}{2}=0.25$



$i=0,1,2$

$j=0,1$

$I_{2j}=0$

$j=1$

$I_{2j-1}=I_1 = \dfrac{0.15}{3}\Big[4f(x_0,y_1)+16f(x_1,y_1) + 8f(x_2,y_1)+16f(x_3,y_1)+f(x_4,y_1)\Big]$
odd

$j=1=m$
even

$I_{2m}=I_2=\dfrac{0.15}{3}\Big[f(x_0,y_2)+4f(x_1,y_2)+2f(x_2,y_2)+4f(x_3,y_2)+f(x_4,y_2)\Big].$

Then,

$$\int_{1.4}^{2}\int_{1}^{1.5}\ln(x+2y)\,dy\,dx \simeq \frac{(0.25)(0.15)}{9}\sum_{\ell=0}^{4}\sum_{p=0}^{2} w_{\ell p}\ln(x_\ell+2y_p)$$

(as in book page 240 10th Ed.)

15 function evaluations