6.1 Linear Systems of Equation. Gaussian Elimination

Examples: Consider the nonhomogeneous systems 2×2.

A $\begin{cases} x_1 + 2x_2 = 1 \\ 3x_1 + x_2 = 2 \end{cases}$ B $\begin{cases} x_1 + 2x_2 = 1 \\ 3x_1 + 6x_2 = 3 \end{cases}$

Unique soln: $(3/5, 1/5)$

Inf. many solns. $(-1,1), (-3,2), \ldots$

C $\begin{cases} x_1 + 2x_2 = 1 \\ 3x_1 + 6x_2 = 2 \end{cases}$ No soln. !

Gauss Elimination:

$$\text{A} \quad \begin{array}{c} E_1 \\ E_2 \end{array} \underbrace{\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 1 & 2 \end{array}\right)}_{= \tilde{A}^{(1)}} \xrightarrow[E_2 - 3E_1 \to E_2]{\substack{\text{row operations} \\ E_1 \to E_1}} \underbrace{\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -5 & -1 \end{array}\right)}_{= \tilde{A}^{(2)}}$$

↳ echelon form

This matrix represent a triang. System:

$$\begin{cases} x_1 + 2x_2 = 1 \\ \quad\ -5x_2 = -1 \end{cases}$$

Backward substitution:

$x_2 = \frac{1}{5}$, $x_1 =$

(B) Augmented matrix.

$$\tilde{A}^{(1)} = \begin{array}{c} E_1 \\ \\ E_2 \end{array} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ & & \\ 3 & 6 & 3 \end{array} \right) \underset{\frown}{\xrightarrow{\substack{E_1 \to E_1 \\ E_2 - 3 * E_1 \to E_2}}} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ & & \\ 0 & 0 & 0 \end{array} \right) = \tilde{A}^{(2)}$$

Equivalent tridiag. syst.: $\begin{cases} x_1 + 2x_2 = 1 \\ \quad 0 = 0 \end{cases}$

$\Rightarrow \quad x_1 = 1 - 2x_2 \rightarrow$ free variable! $\Rightarrow$ Inf. many solns.

(C)

$$\tilde{A}^{(1)} = \left(\begin{array}{cc|c} 1 & 2 & 1 \\ & & \\ 3 & 6 & 2 \end{array} \right) \xrightarrow[E_2 - 3 * E_1 \to E_2]{E_1 \to E_1} \tilde{A}^{(2)} = \left(\begin{array}{cc|c} 1 & 2 & 1 \\ & & \\ 0 & 0 & -1 \end{array} \right)$$

Equivalent to Linear syst. $\begin{cases} x_1 + 2x_2 = 1 \\ \quad 0 = -1 \end{cases}$

Elementary row operations in Gaussian Elimination.

1) Adding a multiple of one row to another row:

$$E_i + \lambda E_j \longrightarrow E_i.$$

2) Multiplying the whole row by a scalar λ

$$\lambda E_i \longrightarrow E_i.$$

3) Interchanging rows.

$$E_i \longleftrightarrow E_j.$$

Consider

$$E_1: \; 2x_1 + 3x_2 + 5x_3 = 23$$

$$E_2: \; 4x_1 + x_2 + 2x_3 = 12$$

$$E_3: \; 6x_1 - x_2 + x_3 = 7$$

Row Reduction:

pivot element

$$\tilde{A}^{(1)} = \left(\begin{array}{ccc|c} 2 & 3 & 5 & 23 \\ 4 & 1 & 2 & 12 \\ 6 & -1 & 1 & 7 \end{array}\right)$$

Step 1: Column 1

$m_{21} = \frac{4}{2} = 2$ 1 M/D

$E_2 - m_{21} E_1 \rightarrow E_2$ 3 M/D, 3 A/S

$m_{31} = \frac{6}{2} = 3$ 1 M/D

$E_3 - m_{31} E_1 \longrightarrow E_3$ 3 M/D, 3 A/S

$$\tilde{A}^{(2)} = \left(\begin{array}{ccc|c} 2 & 3 & 5 & 23 \\ 0 & -5 & -8 & -34 \\ 0 & -10 & -14 & -62 \end{array}\right)$$

Rmk: Not counting multiplication by diag. entry a_{11}, and not considering addition in 1st column.

Step 2: Column 2

$$\tilde{A}^{(2)} = \begin{pmatrix} 2 & 3 & 5 & 23 \\ 0 & -5 & -8 & -34 \\ 0 & -10 & -14 & -62 \end{pmatrix}$$

$m_{32} = \frac{-10}{-5} = 2$ 1 M/D

$E_3 - m_{32} E_2 \rightarrow E_3$

2 M/D, 2 A/S.

$$\tilde{A}^{(3)} = \begin{pmatrix} 2 & 3 & 5 & 23 \\ 0 & -5 & -8 & -34 \\ 0 & 0 & 2 & 6 \end{pmatrix}$$

row echelon form

Equiv. to triang. system

Total Operations:

$$M/D = \underbrace{2 + 2*3}_{i=1} + \underbrace{1 + 1*2}_{i=2} = 11$$

$$A/S = \underbrace{2*3}_{i=1} + \underbrace{1*2}_{i=2} = 8$$

Backward Substitution:

$$\begin{cases} 2x_1 + 3x_2 + 5x_3 = 23 \\ \quad -5x_2 - 8x_3 = -34 \\ \quad\quad 2x_3 = 6 \end{cases}$$

$$X_3 = 6/2 = 3 \qquad (1\ M/D)$$

$$X_2 = \frac{-34-(-8X_3)}{-5} = \frac{-34+24}{-5} = \frac{-10}{-5} = 2 \qquad \begin{pmatrix} 2\ M/D \\ 1\ A/S \end{pmatrix}$$

$$X_1 = \frac{23-(3X_2+5X_3)}{2} = \frac{23-(6+15)}{2} = \frac{2}{2} = 1 \qquad \begin{pmatrix} 3\ M/D \\ 2\ A/S \end{pmatrix}$$

<u>Total operations in backward substitution:</u>

$$M/D = 1+2+3 = 6$$

$$A/S = 0+1+2 = 3$$

<u>Total operations</u> : <u>backward substitution + Gaussian Elim.</u>

$$M/D = 11+6 = 17$$

$$A/S = 8+3 = 11$$

$$A\vec{x} = \vec{b}$$

$$E_1: \ a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = a_{1,n+1}$$

$$E_2: \ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = a_{2,n+1}$$

$$\vdots$$

$$E_n: \ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = a_{n,n+1}$$

$$[U|\vec{c}] = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \dots & a_{1n}^{(1)} & a_{1,n+1}^{(1)} \\ 0 & a_{22}^{(2)} & \dots & a_{2n}^{(2)} & a_{2,n+1}^{(2)} \\ 0 & \ddots & a_{ii}^{(i)} \dots & a_{in}^{(i)} & a_{i,n+1}^{(i)} \\ \vdots & & & \ddots & \\ 0 & \dots & \dots & 0 & a_{nn}^{(n)} \end{bmatrix}$$

At step i

$$A^{(i)} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \dots & \dots & a_{1n}^{(1)} & a_{1,n+1}^{(1)} \\ & \ddots & & & & \\ & & a_{ii}^{(i)} & \overbrace{a_{i,i+1}^{(i)} \ \dots \ a_{in}^{(i)} \quad a_{i,n+1}^{(i)}}^{n-i+1} & & \\ & & n-i \left\{ \begin{matrix} a_{i+1,i}^{(i)} \\ \vdots \\ a_{n,i}^{(i)} \end{matrix} \right. & \dots \dots & a_{n,n}^{(i)} & a_{n,n+1}^{(i} \end{bmatrix}$$

Counting Operations:

Gaussian Elim.

For i fixed $\quad i = 1, \dots, n-1$ (columns).

— Multipl. and divisions to form $m_{ji} = \dfrac{a_{ji}}{a_{ii}}$

$j = i+1, \dots, n$

$= n-i$ (See matrix).

— M/D to form: $E_j - m_{ji} E_i \to E_j$

$j = i+1, \dots, n.$

$$\underset{\text{entries in row}}{(n-i+1)} * \underset{\text{rows}}{(n-i)}$$

— Total $\qquad$ M/D.

$$= \sum_{i=1}^{n-1} (n-i) + (n-i)(n-i+1)$$

$$= \sum_{i=1}^{n-1} (n-i)(n-i+1+1) = \sum_{1}^{n-1} (n-i)(n-i+2) =$$

$$= \sum_{1}^{n-1} \left((n-i)^2 + 2n - 2i\right) = \sum_{1}^{n-1} n^2 - 2ni + i^2 + 2n - 2i$$

$$= (n^2+2n) \sum_{i=1}^{n-1} 1 - (2n+2) \sum_{1}^{n-1} i + \sum_{1}^{n-1} i^2$$

$$= \underbrace{(n^2+2n)}_{n(n+2)}(n-1) - (2n+2)\frac{(n-1)n}{2} + \frac{(n-1)n(2n-1)}{6}$$

$$\boxed{\begin{aligned} \sum_{i=1}^{n} i &= \frac{n(n+1)}{2} \\ \sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} \end{aligned}}$$

$$= \frac{n(n-1)}{6}\left[6(n+2) - 3(2n+2) + 2n - 1\right]$$

$$= \frac{n(n-1)}{6}\left[\cancel{6n} + 12 - \cancel{6n} - 6 + 2n - 1\right]$$

$$= \frac{n(n-1)}{6}\left[2n+5\right] = \frac{2n^3+3n^2-5n}{6}$$

$$\therefore \boxed{\text{TOTAL M/D} = \frac{2n^3+3n^2-5n}{6}}$$

Similarly, it's easy to prove

$$\boxed{\text{TOTAL A/S} = \frac{n^3-n}{3}}$$

Start with $\sum_{i=1}^{n-1} \underbrace{(n-i)}_{\text{rows}} \underbrace{(n-i+1)}_{\text{Add/s.}}$

In particular, in our example 3×3

$$M/D = \frac{54+27-15}{6} = 11 \checkmark$$

$$A/S = \frac{27-3}{3} = 8 \checkmark$$

Gaussian Elim. with Backward Substitution

Consider the Lin. nohomog. Syst. $A\vec{x} = \vec{b}$ (1)

Theorem.– 1) $A_{n\times n}$ non singular.

$\Rightarrow$ there exists U upper triang. and a vector $\vec{c}$, such that

$U\vec{x} = \vec{c}$ is equivalent to (1).

(same set of solns.).

Proof.– It's by construction algorithm. 6.1 in Burden

Gauss-Elim. process:

For $i = 1, .., n-1$

s.1) Find p smallest integer $i \le p \le n$

Such that $a_{pi} \ne 0$ (exists A is nonsing)

S.2) $p \ne i \rightarrow E_p \leftrightarrow E_i$

S.3) For $j = i+1 \dots n$

$$m_{ji} = a_{ji}/a_{ii}$$

$$E_j - m_{ji} E_i \rightarrow E_j$$

end

end

Backward Subst. process:

$a_{nn} \ne 0$ (If not A is sing.)

$x_n = a_{n,n+1}/a_{nn}$

For $i = n-1, .., 1$

$$x_i = \frac{a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}}$$

end.

Considering that these steps 5 and 6 are included in a loop $i = 1.., n-1$

we find

TOTAL M/D and A/S in Gauss. Elim. ~~without the indep. term~~.

$$\sum_{i=1}^{m} i = \frac{m(m+1)}{2} \qquad \sum_{i=1}^{m} i^2 = \frac{m(m+1)(2m+1)}{6}$$

M/D:
$$\sum_{i=1}^{n-1} (n-i)(n-i+2) = \sum_{i=1}^{n-1} n^2 - 2ni + 2n + i^2 - 2i$$

$$= n^2+2n \sum_{i=1}^{n-1} 1 - (2n+2) \sum_{i=1}^{n-1} i + \sum_{i=1}^{n-1} i^2$$

$$= (n^2+2n)(n-1) - (2n+2)\frac{(n-1)n}{2} + \frac{(n-1)n(2n-1)}{6} = \boxed{\frac{2n^3+3n^2-5n}{6}}$$

A/S:
$$\sum_{i=1}^{n-1} (n-i)(n-i+1) = (n^2+n)\sum_{i=1}^{n-1} 1 - (2n+1)\sum_{i=1}^{n-1} i + \sum_{i=1}^{n-1} i^2$$

$$= (n^2+n)(n-1) - (2n+1)\frac{n(n-1)}{2} + \frac{(n-1)n(2n-1)}{6} = \boxed{\frac{n^3-n}{3}}$$

Checking this formula against our example for $n=3$

M/D: $\frac{54+27-15}{6} = \frac{66}{6} = 11$ ✓

A/S: $\frac{27-3}{3} = \frac{24}{3} = 8$ ✓

AS IN BOOK.

Backw. Subst.

step 8: 1 M/D.

Step 9: $\left.\begin{array}{l}(n-i)\ M/D \\ (n-i-1)\ A/S\end{array}\right\}\quad \sum_{j=i+1}^{n} a_{ij}x_j$

$$\frac{a_{i,n+1} - \sum}{a_{ii}} \left\{\begin{array}{l}1\ A/S \\ 1\ M/D.\end{array}\right.$$

$\therefore$ ~~[illegible]~~ Considering loop $i=1,..,n-1$

n=3

$$M/D: \quad 1 + \sum_{i=1}^{n-1} [(n-i)+1] = \frac{n^2+n}{2}$$

6 ✓

3 ✓

$$A/S: \quad \sum_{i=1}^{n-1} [(n-i-1)+1] = \frac{n^2-n}{2}$$

Combining Gauss elim. with Backw subst.

$$\sum_{1}^{n-1} n - \sum i + \sum 1 = n(n-1) - \frac{(n-1)n}{2} + n-1$$

$$= \frac{n(n-1)}{2} + n-1 = (n-1)\left(1+\frac{n}{2}\right) = \frac{n^2+n-2}{2}$$

$$\therefore 1 + \sum = (n^2+n)/2 \checkmark$$

$$\sum n-i = n(n-1) - \frac{(n-1)n}{2} = \frac{1}{2}n(n-1) = \frac{n^2-n}{2} \checkmark$$

M/D:	$\frac{n^3}{3} + n^2 - \frac{n}{3}$
A/S:	$\frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6}$

So # M/D $= \mathcal{O}\left(\frac{n^3}{3}\right)$

and # A/S $= \mathcal{O}\left(n^3/3\right)$

If $n=100$

$$\# M/D \simeq \frac{100^3}{3} = 333.000$$

$$M/D: \frac{2n^3+3n^2-5n}{6} + \frac{n^2}{2} + \frac{n}{2} = \frac{n^3}{3} + n^2 - \frac{n}{3} \checkmark$$

A/S

$$\frac{n^3}{3} - \frac{n}{3} + \frac{n^2}{2} - \frac{n}{2} =$$

$$n^3/3 + n^2/2 - \frac{5}{6}n \checkmark$$