

6.5 Iterative Solution of Linear Equations.

(Burden-Faires)

$$\begin{cases} 2x_1 - x_2 = 1 \\ -x_1 + 3x_2 - x_3 = 8 \\ -x_2 + 2x_3 = -5 \end{cases} \Leftrightarrow A\vec{x} = \vec{b} \quad (1.1)$$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ -5 \end{pmatrix} \Leftrightarrow A\vec{x} = \vec{b}$$

$$\begin{cases} x_1 = \frac{1}{2}x_2 + \frac{1}{2} \\ x_2 = \frac{1}{3}x_1 + \frac{1}{3}x_3 + \frac{8}{3} \\ x_3 = \frac{1}{2}x_2 - \frac{5}{2} \end{cases} \quad (1.2)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{8}{3} \\ -\frac{5}{2} \end{pmatrix}$$

$$\vec{x} = T\vec{x} + \vec{c} \quad (1.3)$$

We are replacing $A\vec{x} = \vec{b}$ by the equivalent system $\vec{x} = T\vec{x} + \vec{c}$

An alternative way to look at the previous linear system is as

$$\left[\begin{matrix} \overset{=D}{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}} - \begin{matrix} \overset{=L}{\begin{pmatrix} 0 & 0 & 0 \\ +1 & 0 & 0 \\ 0 & +1 & 0 \end{pmatrix}} - \begin{matrix} \overset{=U}{\begin{pmatrix} 0 & +1 & 0 \\ 0 & 0 & +1 \\ 0 & 0 & 0 \end{pmatrix}} \end{matrix} \right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 1 \\ 8 \\ -5 \end{pmatrix} \quad (1.4)$$

or briefly

$$A\vec{x} = \vec{b} \Leftrightarrow (D - L - U)\vec{x} = \vec{b}.$$

We can now write: $D\vec{x} = (L + U)\vec{x} + \vec{b}. \quad (1.5)$

In our case,

$$\begin{cases} 2x_1 = x_2 + 1 \\ 3x_2 = x_1 + x_3 + 8 \\ 2x_3 = x_2 - \frac{5}{2} \end{cases}$$

Where

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$L = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Now, D is an invertible matrix because

$$\det(D) = 12 \neq 0.$$

Then, we can write (1.5) as

$$\vec{x} = D^{-1}(L+U)\vec{x} + D^{-1}\vec{b}. \quad (1.6)$$

which in our example corresponds to system $\left. \begin{matrix} (1.2) \\ (1.3) \end{matrix} \right\}$.

Therefore, if we define

$$\boxed{T = D^{-1}(L+U)} \quad \text{and} \quad \boxed{\vec{C} = D^{-1}\vec{b}}$$

(1.6) reduces to

$$\boxed{\vec{x} = T\vec{x} + \vec{C}}$$

Same as (1.3).

This is the equation used by Jacobi to obtain an approximation to the solution $\vec{x}$ of the linear system

$$A\vec{x} = \vec{b}$$

iterating.

So,
for our example, $T = D^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Then,

$$T = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/3 & 0 & 1/3 \\ 0 & 1/2 & 0 \end{pmatrix}$$

and $\boxed{\vec{X} = T\hat{x} + \vec{C}}$

Where $T = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/3 & 0 & 1/3 \\ 0 & 1/2 & 0 \end{pmatrix}$

or

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/3 & 0 & 1/3 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 8/3 \\ -5/2 \end{pmatrix}$$

Jacobi Iteration:

Start with an arbitrary initial approximation

$$\vec{x}^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{pmatrix}$$

and follow with the iterative formula:

$$\vec{x}^{(k)} = T \vec{x}^{(k-1)} + \vec{c}$$

$$\begin{cases} x_1^{(k)} = \frac{1}{2} x_2^{(k-1)} + \frac{1}{2} \\ x_2^{(k)} = \frac{1}{3} x_1^{(k-1)} + \frac{1}{3} x_3^{(k-1)} + \frac{8}{3} \\ x_3^{(k)} = \frac{1}{2} x_2^{(k-1)} - \frac{5}{2} \end{cases}$$

Starting with $\vec{x}^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

We find

$$\vec{x}^{(1)} = (0.5, 2.66, -2.5)^T$$

$$\vec{x}^{(21)} = (2, 3, -1)^T \text{ exact soln.}$$

In general,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1i}x_i + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2i}x_i + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \dots + a_{ii}x_i + \dots + a_{in}x_n = b_i \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{ni}x_i + \dots + a_{nn}x_n = b_n \end{cases}$$

is equivalent to

$$a_{11}x_1 = -a_{12}x_2 - \dots - a_{1i}x_i - \dots - a_{1n}x_n + b_1$$

$$a_{22}x_2 = -a_{21}x_1 - a_{23}x_3 - \dots - a_{2i}x_i - \dots - a_{2n}x_n + b_2$$

$$\vdots$$

$$a_{ii}x_i = -a_{i1}x_1 - \dots - a_{i,i-1}x_{i-1} - a_{i,i+1}x_{i+1} - \dots - a_{in}x_n + b_i$$

$$\vdots$$

$$a_{nn}x_n = -a_{n1}x_1 - \dots - a_{ni}x_i - \dots - a_{n,n-1}x_{n-1} + b_n$$

or in more compact form

$$a_{ii}x_i = - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij}x_j + b_i, \quad i=1, \dots, n. \quad (3.1)$$

$$\text{or } D \vec{x} = (L + U) \vec{x} + \vec{b}.$$

If $a_{ii} \neq 0$

$$x_i = \frac{1}{a_{ii}} \left[- \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} x_j + b_j \right] \quad (4.1)$$

$i=1, \dots, n$

or in matrix form:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix} = - \begin{pmatrix} 0 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \dots & \frac{a_{1i}}{a_{11}} & \dots & \frac{a_{1n}}{a_{11}} \\ \frac{a_{21}}{a_{22}} & 0 & \frac{a_{23}}{a_{22}} & \dots & \frac{a_{2i}}{a_{22}} & \dots & \frac{a_{2n}}{a_{22}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{a_{i1}}{a_{ii}} & \frac{a_{i2}}{a_{ii}} & \dots & \frac{a_{i,i-1}}{a_{ii}} & 0 & \frac{a_{i,i+1}}{a_{ii}} & \dots & \frac{a_{in}}{a_{ii}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{a_{n1}}{a_{nn}} & \frac{a_{n2}}{a_{nn}} & \dots & \dots & \frac{a_{ni}}{a_{nn}} & \dots & \frac{a_{n,n-1}}{a_{nn}} & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} b_1/a_{11} \\ b_2/a_{22} \\ \vdots \\ b_i/a_{ii} \\ \vdots \\ b_n/a_{nn} \end{pmatrix}$$

or

$$\vec{x} = T \vec{x} + \vec{c} \quad \text{or} \quad \vec{x} = D^{-1}(L+U) \vec{x} + D^{-1} \vec{b} \quad (4.2)$$

Comments about existence and uniqueness of Solutions.

i) The matrix A in (1.1) is strictly diagonally dominant then, (1.1) has a unique soln.

ii) Also, because $a_{ii} \neq 0$, $i=1,2,3$. (equivalent to D^{-1} exists)
 T_f can be formed and the two systems

$$A\vec{x} = \vec{b} \quad \text{and} \quad \vec{x} = T_f \vec{x} + \vec{c}$$

are equivalent. It means that they have the same solutions.

iii) Therefore, (1.1) and (1.3) have the same unique solution.

iv) Also, notice

$$\|T\|_{\infty} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} < 1.$$

We will prove later that $\|T\| < 1$ is sufficient condition for the convergence of a general iterative technique:

$$\vec{x}^{(k)} = T \vec{x}^{(k-1)} + \vec{c}.$$

Application to our example:

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ -5 \end{pmatrix}$$

$$A \vec{x} = \vec{b}$$

Jacobi's method leads to $\left\{ \begin{array}{l} \text{dividing matrix } A \text{ by } 3 \\ \text{and splitting this matrix} \\ \text{in two.} \end{array} \right.$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1/2 & 0 \\ 1/3 & 0 & 1/3 \\ 0 & 1/2 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 8/3 \\ -5/2 \end{pmatrix}$$

$$(I - T) \vec{x} = \vec{c}$$

Eigenvalues of matrix T .

$$0 = |T - \lambda I| = \begin{vmatrix} -\lambda & 1/2 & 0 \\ 1/3 & -\lambda & 1/3 \\ 0 & 1/2 & -\lambda \end{vmatrix} = -\lambda^3 + \frac{2}{6}\lambda = \lambda(-\lambda^2 + \frac{1}{3})$$

$$\lambda_1 = 0, \lambda_2 = \frac{1}{\sqrt{3}}, \lambda_3 = -\frac{1}{\sqrt{3}}$$

Then, $\rho(T) = \frac{1}{\sqrt{3}}$

We will prove later: $\rho(T) < 1 \Rightarrow \text{Jacobi converges}$

Jacobi Iteration:

$$\vec{X}^{(k)} = T \vec{X}^{(k-1)} + \vec{C} \quad (5.1)$$

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[- \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} x_j^{(k-1)} + b_i \right] \quad (5.2)$$

$i = 1, \dots, n$

Gauss-Siedel iteration

In our previous example,

$$\begin{cases} x_1^{(k)} = \frac{1}{2} x_2^{(k-1)} + \frac{1}{2} \\ x_2^{(k)} = \frac{1}{3} x_1^{(k)} + \frac{1}{3} x_3^{(k-1)} + \frac{8}{3} \\ x_3^{(k)} = \frac{1}{2} x_2^{(k)} - \frac{5}{2} \end{cases} \quad (5.3)$$

Starting with $\vec{x}^{(0)} = (0, 0, 0)$.

$$\vec{x}^{(1)} = (0.5, 2.8, -1.08)^T$$

$$\vec{x}^{(2)} = (1.9, 2.9, -1.02)^T$$

$$\vdots$$

$$\vec{x}^{(q)} = (2, 3, -1)^T$$

From (5.3)

$$X_1^{(k)} = \frac{1}{2} X_2^{(k-1)} + \frac{1}{2}$$

$$X_2^{(k)} = \frac{1}{3} X_1^{(k)} + \frac{1}{3} X_3^{(k-1)} + \frac{8}{3} =$$

$$= \frac{1}{3} \left[\frac{1}{2} X_2^{(k-1)} + \frac{1}{2} \right] + \frac{1}{3} X_3^{(k-1)} + \frac{8}{3} =$$

$$\therefore X_2^{(k)} = \frac{1}{6} X_2^{(k-1)} + \frac{1}{3} X_3^{(k-1)} + \frac{17}{6}$$

$$X_3^{(k)} = \frac{1}{2} X_2^{(k)} - \frac{5}{2} = \frac{1}{2} \left[\frac{1}{6} X_2^{(k-1)} + \frac{1}{3} X_3^{(k-1)} + \frac{17}{6} \right] - \frac{5}{2}$$

$$\therefore X_3^{(k)} = \frac{1}{12} X_2^{(k-1)} + \frac{1}{6} X_3^{(k-1)} - \frac{13}{12}$$

Therefore,

$$\vec{X}^{(k)} = T_g \vec{X}^{(k-1)} + \vec{C}_g,$$

where

$$T_g = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{6} & \frac{1}{3} \\ 0 & \frac{1}{12} & \frac{1}{6} \end{pmatrix}, \quad \vec{C}_g = \begin{pmatrix} \frac{1}{2} \\ \frac{17}{6} \\ -\frac{13}{12} \end{pmatrix}$$

A more general way to look at Gauss-Siedel.:

$$\begin{cases} 2x_1^{(k)} &= x_2^{(k-1)} + 1 \\ -x_1^{(k)} + 3x_2^{(k)} &= x_3^{(k-1)} + 8 \\ -x_2^{(k)} + 2x_3^{(k)} &= -5 \end{cases}$$

or $(D-L)\vec{x}^{(k)} = U\vec{x}^{(k-1)} + \vec{b}$

$$\Rightarrow \boxed{\vec{x}^{(k)} = (D-L)^{-1}U\vec{x}^{(k-1)} + (D-L)^{-1}\vec{b}}$$

or $\boxed{\vec{x}^{(k)} = T_g\vec{x}^{(k-1)} + \vec{c}_g}$

Where $\boxed{T_g = (D-L)^{-1}U}$ and $\boxed{\vec{c}_g = (D-L)^{-1}\vec{b}}$

In general for any linear system, $A\vec{x} = \vec{b}$ Gauss-Seidel Algorithm
 can be written as (A nonsingular)

$$\begin{aligned}
 a_{11}x_1^{(k)} &= -a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)} - \dots - a_{1n}x_n^{(k-1)} + b_1 \\
 a_{21}x_1^{(k)} + a_{22}x_2^{(k)} &= -a_{23}x_3^{(k-1)} - \dots - a_{2n}x_n^{(k-1)} + b_2 \\
 \vdots &\vdots \\
 a_{i1}x_1^{(k)} + a_{i2}x_2^{(k)} + \dots + a_{ii}x_i^{(k)} &= -a_{i,i+1}x_{i+1}^{(k-1)} - \dots - a_{in}x_n^{(k-1)} + b_i \\
 \vdots &\vdots \\
 a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \dots + a_{nn}x_n^{(k)} &= \dots + b_n
 \end{aligned}$$

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[- \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^n a_{ij}x_j^{(k-1)} \right] + \frac{b_i}{a_{ii}}, \quad i=1, 2, \dots, n$$

→ Formula used for computation.

Theoretical formula:

$$(D-L)\vec{x}^{(k)} = U\vec{x}^{(k-1)} + \vec{b}$$

or

$$\vec{x}^{(k)} = (D-L)^{-1}U\vec{x}^{(k-1)} + (D-L)^{-1}\vec{b}$$

$$\vec{x}^{(k)} = T_g\vec{x}^{(k-1)} + \vec{c}_g$$

Summarizing

Gauss-Seidel.

$$a_{ii} x_i^{(k)} = - \sum_{\substack{j=1 \\ j < i}}^n a_{ij} x_j^{(k)} - \sum_{\substack{j=1 \\ j > i}}^n a_{ij} x_j^{(k-1)} + b_i \quad (6.1)$$

or

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[- \sum_{\substack{j=1 \\ j < i}}^n a_{ij} x_j^{(k)} - \sum_{\substack{j=1 \\ j > i}}^n a_{ij} x_j^{(k-1)} + b_i \right] \quad (6.2).$$

$i = 1, \dots, n$

Successive Overrelaxation (SOR)

Once $\vec{x}_i^{(k)}$ has been obtained from Gauss-Seidel iteration. The convergence can be accelerated by computing a new $x_i^{(k)}$ by means

$$x_i^{(k)} = \omega x_i^{(k)} + (1-\omega) x_i^{(k-1)} \quad (6.3)$$

Where $0 < \omega < 2$

Applying (6.3) to our example. It means

Combining (6.3) and (5.3).

$$\begin{cases} x_1^{(k)} = \omega \left[\frac{1}{2} x_2^{(k-1)} + \frac{1}{2} \right] + (1-\omega) x_1^{(k-1)} \\ x_2^{(k)} = \omega \left[\frac{1}{3} x_1^{(k)} + \frac{1}{3} x_3^{(k-1)} + \frac{8}{3} \right] + (1-\omega) x_2^{(k-1)} \\ x_3^{(k)} = \omega \left[\frac{1}{2} x_2^{(k)} - \frac{5}{2} \right] + (1-\omega) x_3^{(k-1)} \end{cases}$$

In general,

$$x_i^{(k)} = \frac{\omega}{a_{ii}} \left[- \sum_{\substack{j=1 \\ j < i}}^n a_{ij} x_j^{(k)} - \sum_{\substack{j=1 \\ j > i}}^n a_{ij} x_j^{(k-1)} + b_i \right] + (1-\omega) x_i^{(k-1)} \quad (7.1)$$

$i = 1, \dots, n.$

starting with $x^{(0)} = (0, 0, 0)^T$ in our example and $\omega = 1.1$

$$\vec{x}^{(1)} = (0.55, 3.135, -1.026)^T$$

$$\vdots$$

$$\vec{x}^{(7)} = (2, 3, -1) \text{ exact soln.}$$

SOR method in matrix form

$$X_i^{(k)} = \frac{\omega}{a_{ii}} \left[- \sum_{j=1}^{i-1} a_{ij} X_j^{(k)} - \sum_{j=i+1}^n a_{ij} X_j^{(k-1)} + b_i \right] + (1-\omega) X_i^{(k-1)}$$

Then,

$$a_{ii} X_i^{(k)} = -\omega \sum_{j=1}^{i-1} a_{ij} X_j^{(k)} - \omega \sum_{j=i+1}^n a_{ij} X_j^{(k-1)} + \omega b_i + (1-\omega) a_{ii} X_i^{(k-1)}$$

or

$$a_{ii} X_i^{(k)} + \omega \sum_{j=1}^{i-1} a_{ij} X_j^{(k)} = -\omega \sum_{j=i+1}^n a_{ij} X_j^{(k-1)} + \omega b_i + (1-\omega) a_{ii} X_i^{(k-1)}$$

In matrix form in terms of D, L, U .

$$(D - \omega L) \vec{X}^{(k)} = \omega U \vec{X}^{(k-1)} + (1-\omega) D \vec{X}^{(k-1)} + \omega I \vec{b}$$

Therefore,

$$\vec{X}^{(k)} = \overbrace{(D - \omega L)^{-1} [\omega U + (1-\omega) D]}^{= T_\omega} \vec{X}^{(k-1)} + \overbrace{\omega (D - \omega L)^{-1} \vec{b}}^{= \vec{C}_\omega}$$

or

$$\vec{X}^{(k)} = T_\omega \vec{X}^{(k-1)} + \vec{C}_\omega$$

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Summarizing.

Popular iterative algorithm used to approximate

$$A \vec{x} = \vec{b}, \quad \text{where } A = D - L - U.$$

Can be written as

diag. ↓
Lower Triang.
upper triang.

(I) Jacobi Method:

$$D \vec{x}^{(k)} = (L + U) \vec{x}^{(k-1)} + \vec{b}.$$

$$Q = D$$

(II) Gauss-Seidel Method:

$$(D - L) \vec{x}^{(k)} = U \vec{x}^{(k-1)} + \vec{b}.$$

$$Q = D - L$$

(III) SOR METHOD:

$$(D - \omega L) \vec{x}^{(k)} = [\omega U + (1 - \omega) D] \vec{x}^{(k-1)} + \omega \vec{b}$$

or

$$\left(\frac{D}{\omega} - L\right) \vec{x}^{(k)} = \left[U + \frac{1 - \omega}{\omega} D\right] \vec{x}^{(k-1)} + \vec{b}.$$

$$\vec{x} = T \vec{x} + \vec{c}$$

They can also be written as fixed point problems:

(I) Jacobi: $\vec{x}^{(k)} = T_J \vec{x}^{(k-1)} + \vec{c}_J,$

$$T_J \equiv D^{-1}(L + U), \quad \vec{c}_J = D^{-1} \vec{b}.$$

(II) G-S.: $\vec{x}^{(k)} = T_{GS} \vec{x}^{(k-1)} + \vec{c}_{GS},$

$$T_{GS} \equiv (D - L)^{-1} U, \quad \vec{c}_{GS} = (D - L)^{-1} \vec{b}$$

(III) SOR: $\vec{x}^{(k)} = T_{\omega} \vec{x}^{(k-1)} + \vec{c}_{\omega},$

$$T_{\omega} \equiv (D - \omega L)^{-1} [\omega U + (1 - \omega) D]$$

$$\vec{c}_{\omega} \equiv \omega (D - \omega L)^{-1} \vec{b}.$$

Summary 2

$$A\vec{x} = \vec{b}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1i} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2i} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & & & a_{ii} & \dots & a_{in} \\ \vdots & & & \vdots & \ddots & \vdots \\ a_{n1} & & & & & a_{nn} \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & & & & \\ & \ddots & & & \\ & & a_{ii} & & \\ & & & \ddots & \\ & & & & a_{nn} \end{pmatrix} + \begin{pmatrix} 0 & & & & \\ a_{21} & 0 & & & \\ \vdots & \vdots & \ddots & & \\ a_{n1} & & & 0 & \end{pmatrix} + \begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ & \ddots & & \vdots \\ 0 & & & a_{n-1,n} \\ & & 0 & \end{pmatrix}$$

$\parallel$ $\parallel$
 $-L$ $-U$

or $A = D$

or $A\vec{x} = \vec{b} \Leftrightarrow D\vec{x} = (L+U)\vec{x} + \vec{b}$

or $A\vec{x} = \vec{b} \Leftrightarrow (D-L)\vec{x} = U\vec{x} + \vec{b}$

For Computation

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[- \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij} x_j^{(k-1)} + b_i \right]$$

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[- \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^n a_{ij} x_j^{(k-1)} + b_i \right]$$

$$x_i^{(k)} = \frac{\omega}{a_{ii}} \left[- \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^n a_{ij} x_j^{(k-1)} + b_i \right] + (1-\omega) x_i^{(k-1)}$$

For Analysis

$$\vec{x} = D^{-1}(L+U)\vec{x} + D^{-1}\vec{b}, \quad a_{ii} \neq 0$$

or $\boxed{\vec{x} = \underset{||}{T_j} \vec{x} + \underset{||}{\vec{c}_j}}$

$$(D-L)\vec{x} = U\vec{x} + \vec{b}, \quad \text{if } D-L \text{ invertible}$$

$$\boxed{\vec{x} = \underset{||}{(D-L)^{-1}U} \vec{x} + \underset{||}{(D-L)^{-1}\vec{b}}}$$

or $\vec{x} = \underset{||}{T_{GS}} \vec{x} + \underset{||}{\vec{c}_{GS}}$

$$(D-\omega L)\vec{x} = \omega U\vec{x} + (1-\omega)D\vec{x} + \omega I\vec{b}$$

If $D-\omega L$ is invertible

$$\boxed{\vec{x} = \underset{||}{(D-\omega L)^{-1}[\omega U + (1-\omega)D]} \vec{x} + \underset{||}{\omega (D-\omega L)^{-1}\vec{b}}}$$

or $\vec{x} = \underset{||}{T_\omega} \vec{x} + \underset{||}{\vec{c}_\omega}$

The condition $0 < \omega < 2$ is also sufficient for special set of matrices A .

Thm. (Ostrowski-Reich).

If A is positive definite and $\omega \in (0, 2)$ then, SOR converges for any choice of initial approximation $\vec{u}^{(0)}$.

Recall A is positive definite if $\vec{x}^T A \vec{x} > 0$ for any $\vec{x} \neq \vec{0}$, and A is symmetric.

Also, A is P.D. $\Leftrightarrow$ All eigenvalues of A are positive.

Thm. If A is P.D. and tridiagonal, then $\rho(M_\omega) = \rho^2(M_j)^{\frac{1}{2}}$ and the optimal choice of ω for SOR method is

$$\omega_{\text{opt}} = \frac{2}{1 + \sqrt{1 - \rho^2(M_j)}}$$

and $\rho(M_\omega) = \omega - 1$.

Notice $1 < \omega_{\text{opt}} < 2$

Application of previous theorem to our linear system.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

A is P.D since $|2| = 2^2$, $\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} = 5 > 0$
 and $|A| = 12 - 4 = 8 > 0$. and it's also tridiag.
 then the prev. Thm. can be applied to A. **A is Symm.**

$$A = D - L - U = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A\tilde{x} = \tilde{b} \Rightarrow D\tilde{x} = (L+U)\tilde{x} + \tilde{b} \Rightarrow \tilde{x} = D^{-1}(L+U)\tilde{x} + D^{-1}\tilde{b}$$

or $\boxed{\tilde{x} = M_j \tilde{x} + \tilde{C}_j}$, $M_j \equiv D^{-1}(L+U)$.

For our example,

$$M_j = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/3 & 0 & 1/3 \\ 0 & 1/2 & 0 \end{bmatrix}$$

$$\Rightarrow |M_j - \lambda I| = \begin{vmatrix} -\lambda & 1/2 & 0 \\ 1/3 & -\lambda & 1/3 \\ 0 & 1/2 & -\lambda \end{vmatrix} = -\lambda^3 + \frac{2}{6}\lambda = 0 \Rightarrow \lambda(\lambda^2 - \frac{2}{6}) = 0$$

$$\Rightarrow \lambda_1 = 0, \lambda_{2,3} = \pm \frac{\sqrt{2}}{\sqrt{2}\sqrt{3}} = \pm \frac{\sqrt{3}}{3}$$

$$\Rightarrow \rho(M_j) = \frac{\sqrt{3}}{3} \approx 0.58.$$

$$\Rightarrow \text{Optimal } \omega, \quad \omega_{\text{opt}} = \frac{2}{1 + \sqrt{1 - \rho^2(M_j)}} \approx \underline{\underline{1.1}} \quad \checkmark$$

Analysis of Convergence

Consider the fixed point problem:

$$\vec{X} = T\vec{X} + \vec{C}, \quad \vec{X} \in \mathbb{R}^n \quad (5.1)$$

and assume, it has a unique fixed point $\vec{X}^*$.

It means, that there is a $\vec{X}^*$ that satisfies the equation (5.1),

$$\vec{X}^* = T\vec{X}^* + \vec{C}. \quad (5.2)$$

Also, consider the iterative technique corresponding to (5.1) given by

$$\vec{X}^{(k)} = T\vec{X}^{(k-1)} + \vec{C} \quad (5.3)$$

Theorem:

If $\|T\| < 1$, then

i) The fixed point problem (5.1) has a unique fixed point.

ii) The iteration (5.3) converges to the fixed point $\bar{x}^*$ of (5.1).

Proof.

i) Assume $\bar{x}^* \neq \bar{x}^{**}$ and

$$\bar{x}^* = T\bar{x}^* + \bar{c} \quad (5.4)$$

$$\bar{x}^{**} = T\bar{x}^{**} + \bar{c} \quad (5.5)$$

Subtracting (5.5) from (5.4) and using linearity

$$\bar{x}^* - \bar{x}^{**} = T(\bar{x}^* - \bar{x}^{**})$$

Then using any norm in $\mathbb{R}^n$

$$\|\bar{x}^* - \bar{x}^{**}\| = \|T(\bar{x}^* - \bar{x}^{**})\| \leq \|T\| \|\bar{x}^* - \bar{x}^{**}\| < \|\bar{x}^* - \bar{x}^{**}\|$$

Contradiction! . Therefore, $\bar{x}^{**} = \bar{x}^*$.

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ii) Subtracting (5.2) from (5.3), and using linearity leads to

$$\vec{X}^{(k)} - \vec{X}^* = T (\vec{X}^{(k-1)} - \vec{X}^*) \quad (5.4)$$

defining $\vec{e}^k = \vec{X}^{(k)} - \vec{X}^*$

Equ. (5.4) reduces to

$$\vec{e}^{(k)} = T \vec{e}^{(k-1)}$$

Iterating the right term

$$\begin{aligned} \vec{e}^{(k)} &= T \vec{e}^{(k-1)} = T(T \vec{e}^{(k-2)}) = T^2 \vec{e}^{(k-2)} = \dots \\ &= T^k \vec{e}^{(0)} \end{aligned}$$

Thus, Using any norm in $\mathbb{R}^n$

$$\|\vec{e}^{(k)}\| = \|T^k \vec{e}^{(0)}\| \leq \|T^k\| \|\vec{e}^{(0)}\| \leq \|T\|^k \|\vec{e}^{(0)}\|$$

Because $\|T\| < 1 \Rightarrow \|T\|^k \xrightarrow[k \rightarrow \infty]{} 0$

And $\|\vec{e}^{(k)}\| \leq \|T^k\| \|\vec{e}^{(0)}\| \xrightarrow[k \rightarrow \infty]{} 0$

or $\|\vec{X}^{(k)} - \vec{X}^*\| \xrightarrow[k \rightarrow \infty]{} 0 \quad \checkmark$