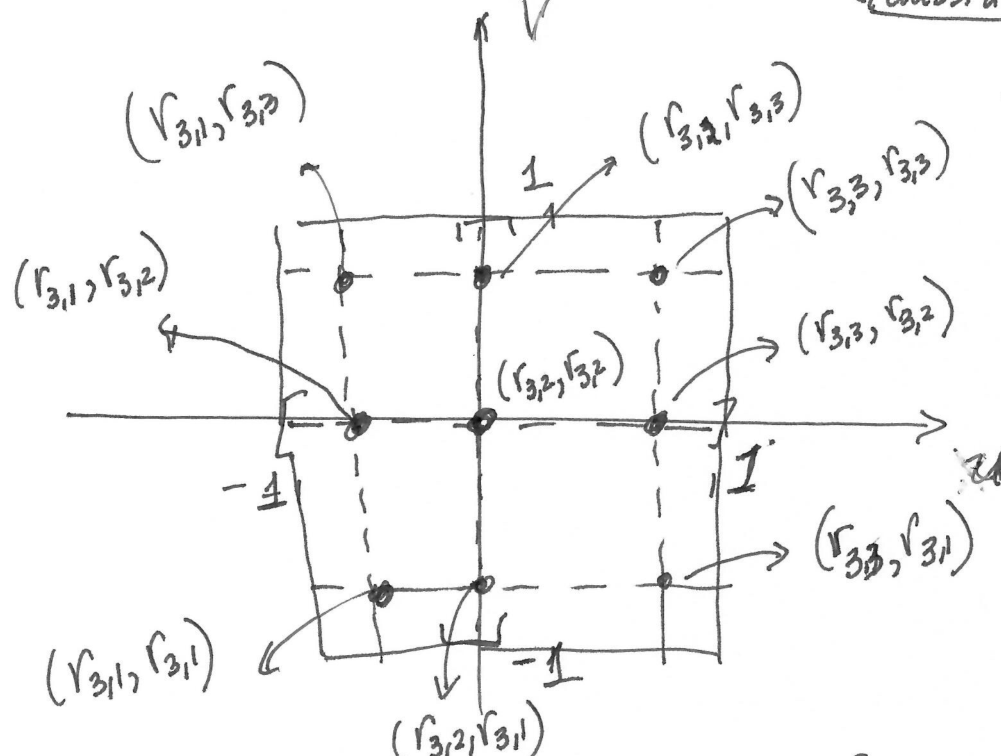
Gaussian Quad

$n=3$



Apply Gaussian quad for this rectangular region with $n=3$ in both directions.

$$\boxed{\int_{-1}^{1}\int_{-1}^{1} f(u,v)\,du\,dv = \sum_{i=1}^{3}\sum_{j=1}^{3} c_{3,i}\, c_{3,j}\, f(r_{3,i}, r_{3,j})} \quad (1.1)$$

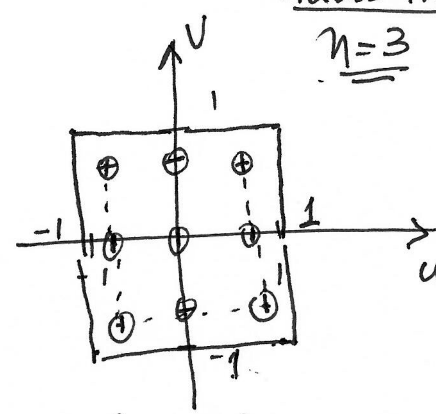
where $r_{3,i}, r_{3,j}$ are the roots of the Legendre polynomial degree 3. ($i, j = 1,2,3$)

and $c_{3,i}, c_{3,j}$ are the corresponding Coefficients using Gauss integration.

Derivation of (1.1).

Gauss integration

$n = 3$



the roots are

$r_{3,1} = -\sqrt{\frac{3}{5}}$, $r_{3,2} = 0$

$r_{3,3} = \sqrt{\frac{3}{5}}$.

$$\int_{-1}^{1}\int_{-1}^{1} g(u,v)\,dv\,du =$$

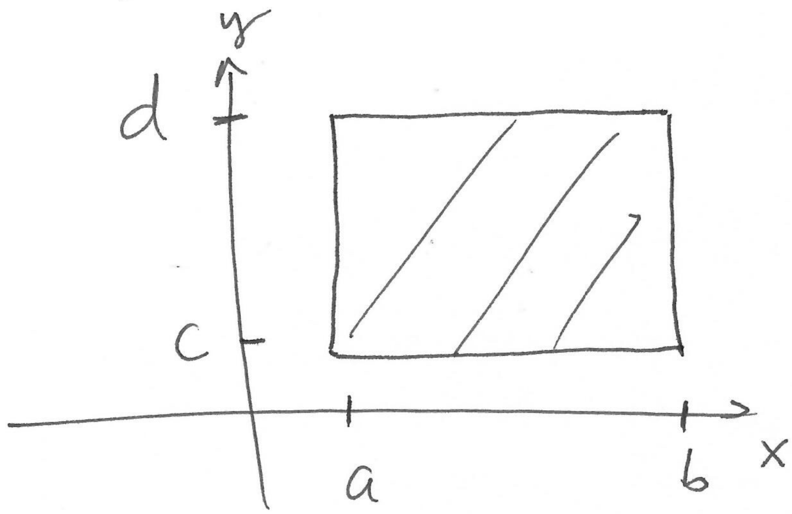
$$= \int_{-1}^{1}\left(\int_{-1}^{1} g(u,v)\,dv\right)du$$

$$= \int_{-1}^{1}\left(\sum_{j=1}^{3} c_{3,j}\, g(u, r_{3,j})\right)du$$

$$= \sum_{j=1}^{3} c_{3j}\int_{-1}^{1} g(u, r_{3,j})\,du = \sum_{j=1}^{3} c_{3j} \sum_{i=1}^{3} c_{3,i}\, g(r_{3,i}, r_{3,j})$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3} c_{3,i}\, c_{3,j}\, g(r_{3,i}, r_{3,j})$$

If the region is an arbitrary rectangle.



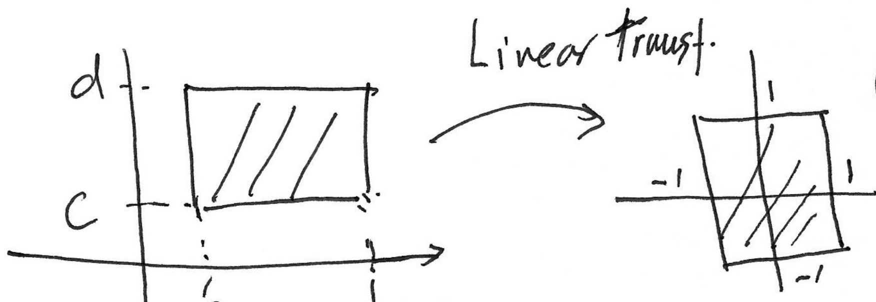
We need to make a transformation to a $[-1,1]\times[-1,1]$ rectangle

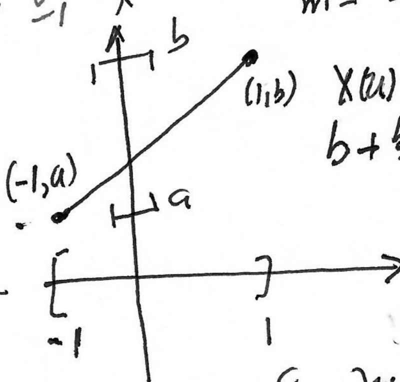
$$x(u) = \frac{1}{2}\left[(b-a)u + b+a\right] \rightarrow dx = \frac{b-a}{2}\,du$$

$$y(v) = \frac{1}{2}\left[(d-c)v + d+c\right] \rightarrow dy = \frac{d-c}{2}\,dv$$

then,

$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \left(\frac{b-a}{2}\right)\left(\frac{d-c}{2}\right)\int_{-1}^{1}\int_{-1}^{1} f(x(u),y(v))\,du\,dv$$



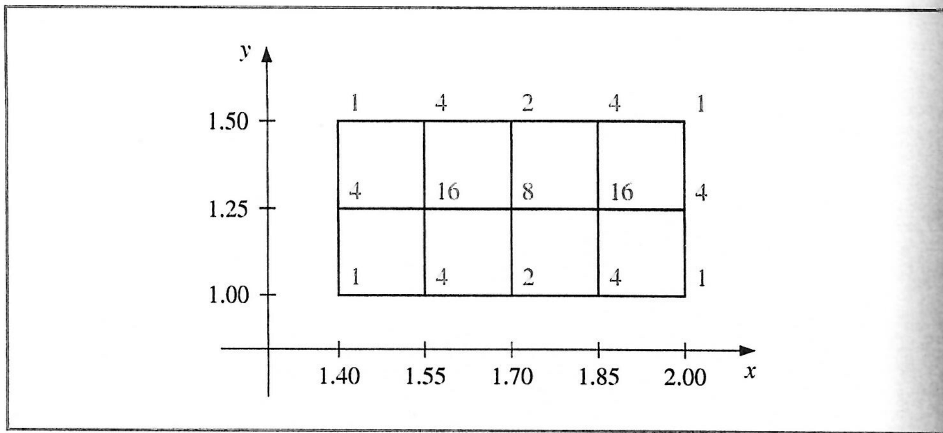


$m = \frac{b-a}{2}$

$X(u) = b + \frac{b-a}{2}(u-1)$

or $X(u) = \frac{(b-a)u + b + a}{2}$

Figure 4.20



Using Simpson
$n = 2$
$m = 1$
in my notation
but
$n = 4$
$m = 2$
in book notation

The approximation is

$$\int_{1.4}^{2.0}\int_{1.0}^{1.5} \ln(x+2y)\,dy\,dx \approx \frac{(0.15)(0.25)}{9}\sum_{i=0}^{4}\sum_{j=0}^{2} w_{i,j}\ln(x_i+2y_j)$$
$$= 0.4295524387.$$

We have

$$\frac{\partial^4 f}{\partial x^4}(x,y) = \frac{-6}{(x+2y)^4} \quad \text{and} \quad \frac{\partial^4 f}{\partial y^4}(x,y) = \frac{-96}{(x+2y)^4},$$

and the maximum values of the absolute values of these partial derivatives occur on R when $x = 1.4$ and $y = 1.0$. So, the error is bounded by

$$|E| \le \frac{(0.5)(0.6)}{180}\left[(0.15)^4 \max_{(x,y)\text{in}R}\frac{6}{(x+2y)^4} + (0.25)^4 \max_{(x,y)\text{in}R}\frac{96}{(x+2y)^4}\right] \le 4.72 \times 10^{-6}$$

The actual value of the integral to 10 decimal places is

$$\int_{1.4}^{2.0}\int_{1.0}^{1.5} \ln(x+2y)\,dy\,dx = 0.4295545265,$$

so the approximation is accurate to within 2.1×10^{-6}.

The same techniques can be applied for the approximation of triple integrals as well as higher integrals for functions of more than three variables. The number of functional evaluations required for the approximation is the product of the number of functional evaluations required when the method is applied to each variable.

Gaussian Quadrature for Double Integral Approximation

To reduce the number of functional evaluations, more efficient methods, such as Gaussian quadrature, Romberg integration, or Adaptive quadrature, can be incorporated in place of the Newton-Cotes formulas. The following example illustrates the use of Gaussian quadrature for the integral considered in Example 1.

Example 2 Use Gaussian quadrature with $n = 3$ in both dimensions to approximate the integral

$$\int_{1.4}^{2.0}\int_{1.0}^{1.5} \ln(x+2y)\,dy\,dx.$$

Solution Before employing Gaussian quadrature to approximate this integral, we need to transform the region of integration

$$R = \{ (x, y) \mid 1.4 \le x \le 2.0, 1.0 \le y \le 1.5 \} \quad \text{into}$$

$$\hat{R} = \{ (u, v) \mid -1 \le u \le 1, -1 \le v \le 1 \}.$$

The linear transformations that accomplish this are

$$u = \frac{1}{2.0 - 1.4}(2x - 1.4 - 2.0) \quad \text{and} \quad v = \frac{1}{1.5 - 1.0}(2y - 1.0 - 1.5),$$

or, equivalently, $x = 0.3u + 1.7$ and $y = 0.25v + 1.25$. Employing this change of variables gives an integral on which Gaussian quadrature can be applied:

$$\int_{1.4}^{2.0} \int_{1.0}^{1.5} \ln(x + 2y)\, dy\, dx = 0.075 \int_{-1}^{1} \int_{-1}^{1} \ln(0.3u + 0.5v + 4.2)\, dv\, du.$$

The Gaussian quadrature formula for $n = 3$ in both u and v requires that we use the nodes

$$u_1 = v_1 = r_{3,2} = 0, \quad u_0 = v_0 = r_{3,1} = -0.7745966692,$$

and

$$u_2 = v_2 = r_{3,3} = 0.7745966692.$$

The associated weights are $c_{3,2} = 0.\overline{8}$ and $c_{3,1} = c_{3,3} = 0.\overline{5}$. (These are given in Table 4.12 on page 232.) The resulting approximation is

$$\int_{1.4}^{2.0} \int_{1.0}^{1.5} \ln(x + 2y)\, dy\, dx \approx 0.075 \sum_{i=1}^{3} \sum_{j=1}^{3} c_{3,i} c_{3,j} \ln(0.3r_{3,i} + 0.5r_{3,j} + 4.2)$$

$$= 0.4295545313.$$

Although this result requires only nine functional evaluations compared to 15 for the Composite Simpson's rule considered in Example 1, it is accurate to within 4.8×10^{-9}, compared to 2.1×10^{-6} accuracy in Example 1. ■

Nonrectangular Regions

The use of approximation methods for double integrals is not limited to integrals with rectangular regions of integration. The techniques previously discussed can be modified to approximate double integrals of the form

$$\int_{a}^{b} \int_{c(x)}^{d(x)} f(x, y)\, dy\, dx \tag{4.42}$$

or

$$\int_{c}^{d} \int_{a(y)}^{b(y)} f(x, y)\, dx\, dy. \tag{4.43}$$

In fact, integrals on regions not of this type can also be approximated by performing appropriate partitions of the region. (See Exercise 10.)

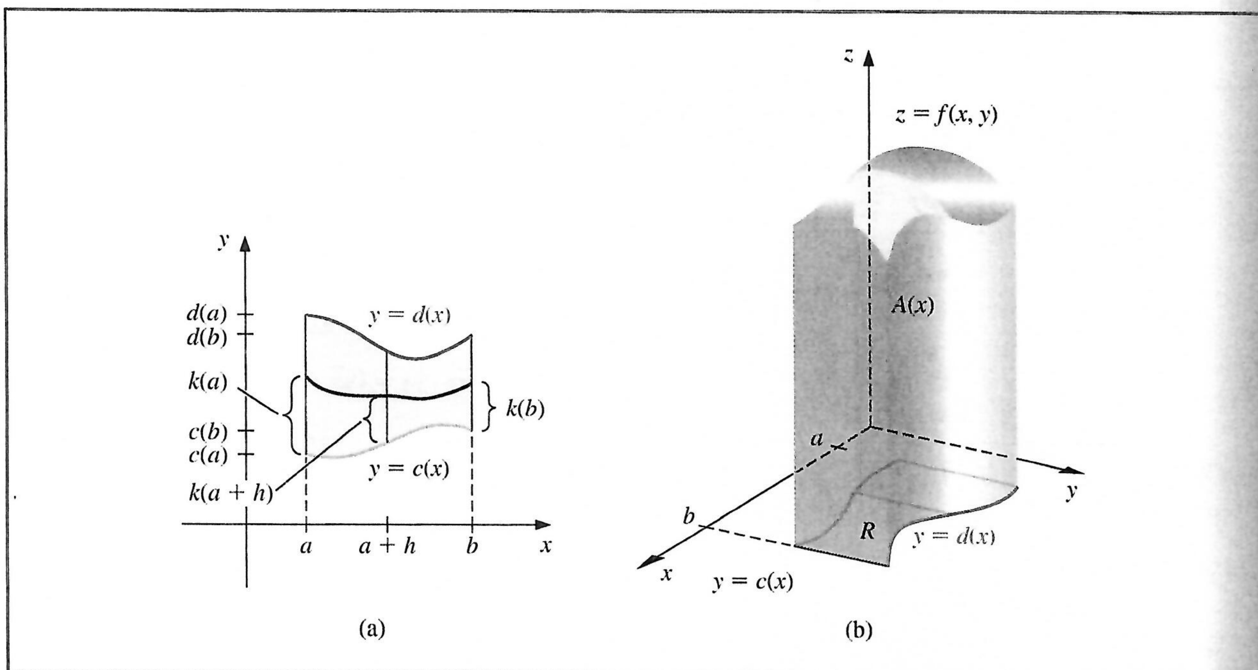
To describe the technique involved with approximating an integral in the form

$$\int_{a}^{b} \int_{c(x)}^{d(x)} f(x, y)\, dy\, dx,$$

we will use the basic Simpson's rule to integrate with respect to both variables. The step size for the variable x is $h = (b - a)/2$, but the step size for y varies with x (see Figure 4.21) and is written

$$k(x) = \frac{d(x) - c(x)}{2}.$$

Figure 4.21



This gives

$$
\begin{aligned}
\int_a^b \int_{c(x)}^{d(x)} f(x, y)\, dy\, dx &\approx \int_a^b \frac{k(x)}{3}[f(x, c(x)) + 4f(x, c(x) + k(x)) + f(x, d(x))]\, dx \\
&\approx \frac{h}{3}\left\{\frac{k(a)}{3}[f(a, c(a)) + 4f(a, c(a) + k(a)) + f(a, d(a))]\right. \\
&\quad + \frac{4k(a + h)}{3}[f(a + h, c(a + h)) + 4f(a + h, c(a + h) \\
&\quad + k(a + h)) + f(a + h, d(a + h))] \\
&\quad \left. + \frac{k(b)}{3}[f(b, c(b)) + 4f(b, c(b) + k(b)) + f(b, d(b))]\right\}
\end{aligned}
$$

$\int_a^b \frac{k(x)}{3} f(x, c(x))\, dx$
Simpson appx.

Algorithm 4.4 applies the Composite Simpson's rule to an integral in the form (4.42) Integrals in the form (4.43) can, of course, be handled similarly.