4.7 Gaussian Quadrature

In general, all quad. rules are a particular case of

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i) \qquad (1)$$

Important question: Best choices of c_i's and x_i's for higher degree of precision.

For example, for $n=2$: $\underline{x_1, x_2}$

and $[a,b]=[-1,1]$, what should be x_1, x_2, c_1, c_2

such that

$$\int_{-1}^{1} f(x)\,dx \simeq c_1 f(x_1) + c_2 f(x_2) \qquad (2)$$

is of highest degree of precision?

Since we want to determine 4 unknowns, it might be that degree of precision for an optimal choice is 3. It means we will try to determine x_1, x_2, c_1, c_2 such that (2) is exact for $1, x, x^2$, and x^3.

We arrive to the following system of equations:

$$\begin{cases} c_1 + c_2 = \int_{-1}^{1} dx = 2 \\ c_1 x_1 + c_2 x_2 = \int_{-1}^{1} x\,dx = 0 \\ c_1 x_1^2 + c_2 x_2^2 = \int_{-1}^{1} x^2 dx = \left.\frac{x^3}{3}\right|_{-1}^{1} = \frac{2}{3} \\ c_1 x_1^3 + c_2 x_2^3 = \int_{-1}^{1} x^3 dx = 0 \end{cases}$$

Nonlinear system. Solutions are: $c_1 = 1$, $x_1 = -\frac{\sqrt{3}}{3}$
$c_2 = 1$, $x_2 = \frac{\sqrt{3}}{3}$.

Therefore, quadrature rule using only two nodes with highest degree of precision is

$$\boxed{\int_{-1}^{1} f(x)\,dx \simeq f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right).} \qquad (3)$$

We can generate higher degree of precision formulas by following a similar procedure. But, there is an EASIER WAY.

- Notice that $x_1 = -\frac{\sqrt{3}}{3}$ and $x_2 = \frac{\sqrt{3}}{3}$ are the two roots of $P_2(x) = x^2 - \frac{1}{3}$ (Legendre polyn. degree 2). They gave quad. formula exact to polynomials of degree less than 4.

- Show other Legendre polynomials (MAPLE).

In general, a quadrature formula exact for polynomials of degree less than $2n$ can be obtained using the roots: $x_1, .., x_n$ of Legendre polynomials of degree n. How about the coefficients C_i's ?

<u>Thm 4.7</u> 1) $x_1, x_2, , x_n$ are roots of Legendre polynomial $P_n(x)$

2) The Coefficients C_i, $i=1,, n$ are defined by

$$C_i = \int_{-1}^{1} \left(\prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(x-x_j)}{x_i - x_j} \right) dx$$

Then,

if $P(x)$ is any polynomial of degree less than $2n$

$$\int_{-1}^{1} P(x)dx \underset{\text{"exactly"}}{=} \sum_{i=1}^{n} C_i P(x_i)$$

It means that the formula

$$\int_{-1}^{1} f(x)dx \simeq \sum_{i=1}^{n} C_i f(x_i)$$

is at least of degree of precision $2n-1$.

Therefore, (3) is a Quad. Gauss formula with two nodes ($n=2$) and degree of precision 3 ($2n-1$).

The next formula, by using the previous thm is given by

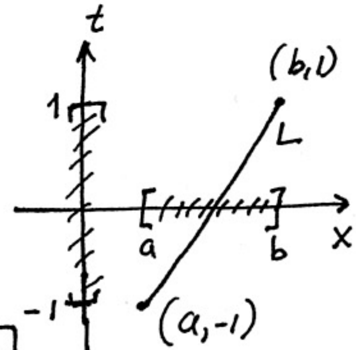
$$\int_{-1}^{1} f(x)dx \simeq \frac{5}{9} f\left(-\sqrt{\tfrac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\tfrac{3}{5}}\right)$$

$$\simeq 0.556\, f(-0.77) + 0.889\, f(0) + 0.556\, f(0.77)$$

with degree of precision $2(3)-1 = 5$.

Gaussian Quadrature on any Interval

Approx $\int_a^b f(x)dx$ using Gauss quad.



I) Transform $[a,b] \longrightarrow [-1,1]$

$x \longrightarrow t(x)$

Slope of L: $m = \frac{2}{b-a}$

L: $t-1 = \frac{2}{b-a}(x-b) \Rightarrow x-b = \left(\frac{b-a}{2}\right)\cdot(t-1)$

$$\boxed{t = \frac{2}{b-a}(x-b)+1}$$

or $\boxed{x = \frac{b-a}{2}(t-1) + b}$, $t \in [-1,1]$

$\Rightarrow \quad dx = \frac{b-a}{2} dt$

Thus, changing variables $x \to t$

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\frac{b-a}{2}(t-1)+b\right)\left(\frac{b-a}{2}\right)dt$$

In particular, for

$$\int_{1=a}^{1.5=b} e^{-x^2}\,dx$$

$$\frac{b-a}{2} = \frac{1}{4}$$

$x=1 \longrightarrow t=-1$

$x=1.5 \rightarrow t=1$

$$x = \frac{b-a}{2}(t-1)+b = \frac{1}{4}(t-1)+\frac{3}{2} = \frac{t-1+6}{4} = \frac{t+5}{4}$$

Thus,

$$\int_1^{1.5} e^{-x^2}dx = \int_{-1}^{1} e^{-\left(\frac{t+5}{4}\right)^2}\frac{1}{4}\,dt \simeq$$

Gauss 2 points

$$\simeq e^{-\frac{\left(-\frac{\sqrt{3}}{3}+5\right)^2}{16}} + e^{-\frac{\left(\frac{\sqrt{3}}{3}+5\right)^2}{16}} \simeq 0.1094003$$

Gauss 3 points

$$\simeq \frac{5}{9}e^{-\frac{\left(-\sqrt{\frac{3}{5}}+5\right)^2}{16}} + \frac{8}{9}e^{-\frac{(0+5)^2}{16}} + \frac{5}{9}e^{-\frac{\left(\sqrt{\frac{3}{5}}+5\right)^2}{16}}$$

$$\simeq 0.1093642$$

Legendre Polynomials

The technique we have described could be used to determine the nodes and coefficients for formulas that give exact results for higher-degree polynomials, but an alternative method obtains them more easily. In Sections 8.2 and 8.3, we will consider various collections of orthogonal polynomials, functions that have the property that a particular definite integral of the product of any two of them is 0. The set that is relevant to our problem is the Legendre polynomials, a collection $\{P_0(x), P_1(x), \ldots, P_n(x), \ldots,\}$ with the following properties:

(1) For each n, $P_n(x)$ is a monic polynomial of degree n.

(2) $\int_{-1}^{1} P(x)\, P_n(x)\, dx = 0$ whenever $P(x)$ is a polynomial of degree less than n.

The first few Legendre polynomials are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = x^2 - \frac{1}{3},$$

$$P_3(x) = x^3 - \frac{3}{5}x, \quad \text{and} \quad P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}.$$

The roots of these polynomials are distinct, lie in the interval $(-1, 1)$, have a symmetry with respect to the origin, and, most important, are the correct choice for determining the parameters that give us the nodes and coefficients for our quadrature method.

The nodes $x_1, x_2, \ldots, x_n$ needed to produce an integral approximation formula that gives exact results for any polynomial of degree less than $2n$ are the roots of the n th-degree exact results for any polynomial of degree less than $2n$ are the roots of the n th-degree Legendre polynomial. This is established by the following result.

Theorem 4.7.

Suppose that $x_1, x_2, \ldots, x_n$ are the roots of the n th Legendre polynomial $P_n(x)$ and that for each $i = 1, 2, \ldots, n$, the numbers c_i are defined by

$$c_i = \int_{-1}^{1} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} \, dx.$$

If $P(x)$ is any polynomial of degree less than $2n$, then

$$\int_{-1}^{1} P(x) \, dx = \sum_{i=1}^{n} c_i P(x_i).$$

[redacted]

Let us first consider the situation for a polynomial $P(x)$ of degree less than n. Rewrite $P(x)$ in terms of $(n-1)$ st Lagrange coefficient polynomials with nodes at the roots of the n th Legendre polynomial $P_n(x)$. The error term for this representation involves the n th derivative of $P(x)$. Since $P(x)$ is of degree less than n, the n th derivative of $P(x)$ is 0, and this representation of is exact. So,

Example 1.

Approximate $\int_{-1}^{1} e^x \cos x \, dx$ using Gaussian quadrature with $n = 3$.

Solution

The entries in Table 4.12 give us

$$\int_{-1}^{1} e^x \cos x \, dx \approx 0.5e^{0.774596692} \cos 0.774596692 + 0.8 \cos 0 + 0.5e^{-0.774596692} \cos(-0.774596692)$$

$$= 1.9333904.$$

Integration by parts can be used to show that the true value of the integral is 1.9334214, so the absolute error is less than 3.2×10^{-5}.

Table 4.12.

n	**Roots** $r_{n,i}$	**Coefficients** $c_{n,i}$
2	0.5773502692	1.0000000000
	−0.5773502692	1.0000000000
3	0.7745966692	0.5555555556
	0.0000000000	0.8888888889
	−0.7745966692	0.5555555556
4	0.8611363116	0.3478548451
	0.3399810436	0.6521451549
	−0.3399810436	0.6521451549
	−0.8611363116	0.3478548451
5	0.9061798459	0.2369268850
	0.5384693101	0.4786286705
	0.0000000000	0.5688888889
	−0.5384693101	0.4786286705
	−0.9061798459	0.2369268850