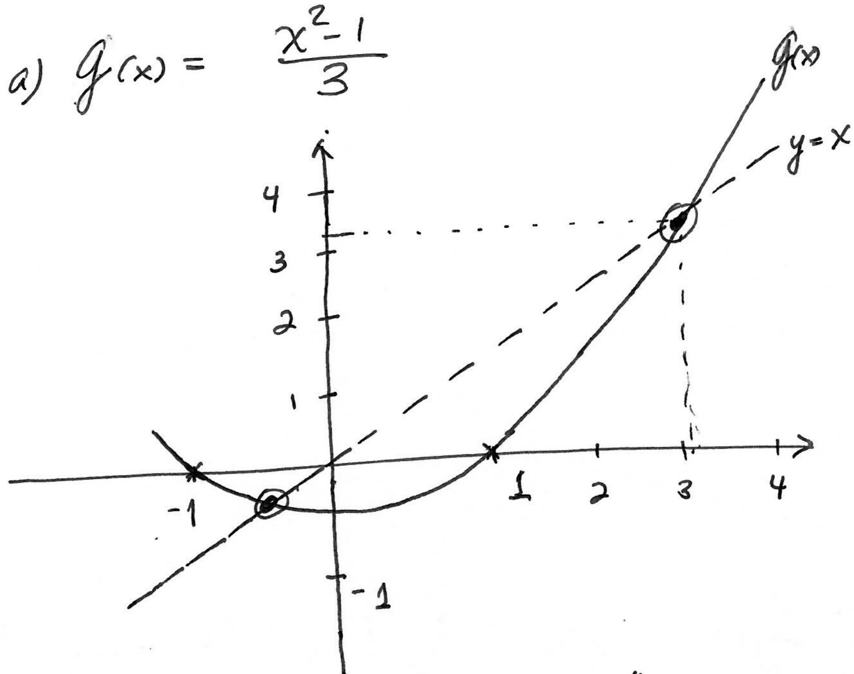
2.2. Fixed Point Iteration.

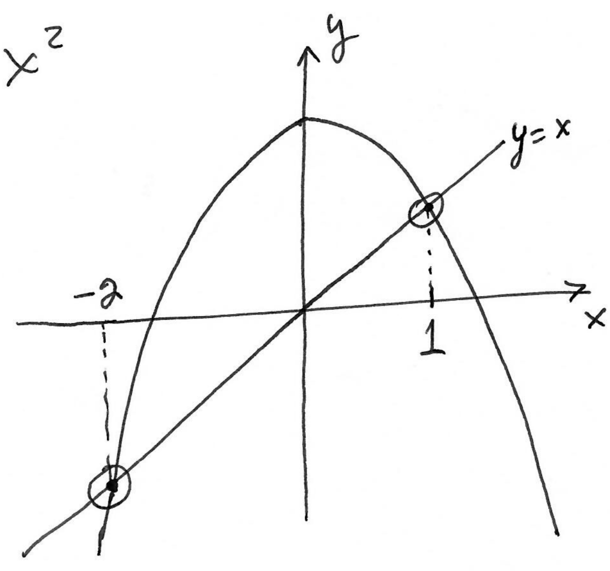
Example: a) $g(x) = \frac{x^2-1}{3}$



This parabola clearly has two "fixed points".

$$g(x) = x.$$

b) $g(x) = 2 - x^2$



$x_1 = -2$

Since

$g(-2) = 2 - (-2)^2 =$
$= 2 - 4 = -2$

$\therefore$ $\boxed{g(-2) = -2}$

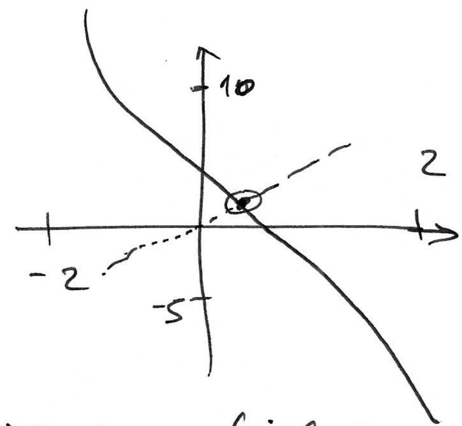
and

$g(1) = 2 - 1^2 = 1$

$\boxed{g(1) = 1}$

c) $g(x) = 2 - e^{x} + x^{2} - 3x$.

Find x s.t. $\underline{g(x) = x}$?



In examples (a) and (b), we can find the fixed points analytically. However in example (c), we will need a Calculator or a Computer.

In (a)

$$g(p) = p$$

$$\Rightarrow \quad \frac{p^2 - 1}{3} = p \quad \Rightarrow \quad p^2 - 3p - 1 = 0$$

$$\Rightarrow \quad p_{1,2} = \frac{3 \pm \sqrt{9+4}}{2} = \frac{1}{2}\left(3 \pm \sqrt{13}\right).$$

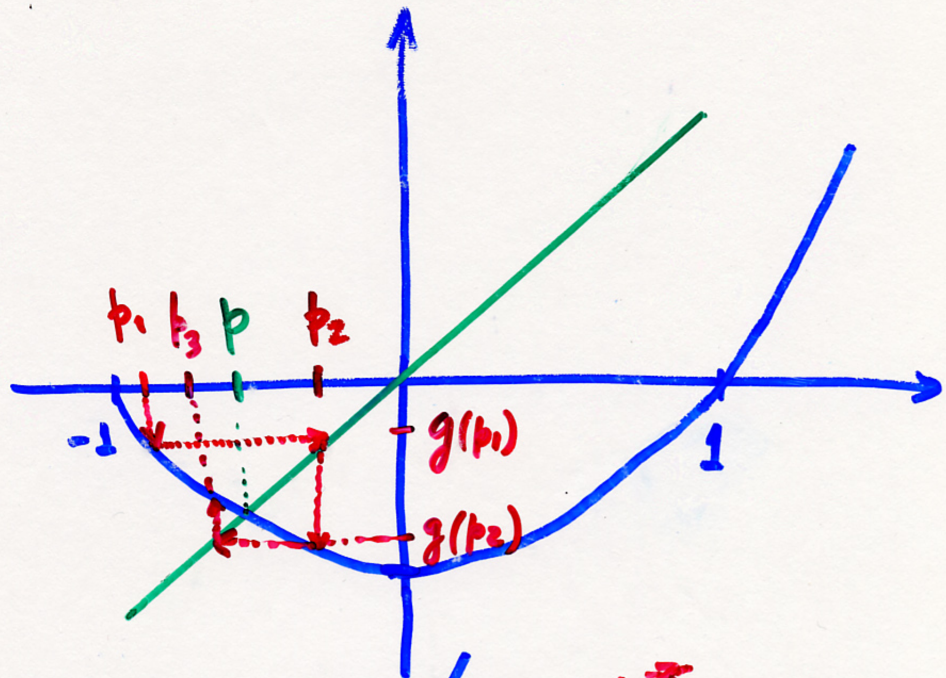
We can also try a "functional iteration" to approximate a fixed point.

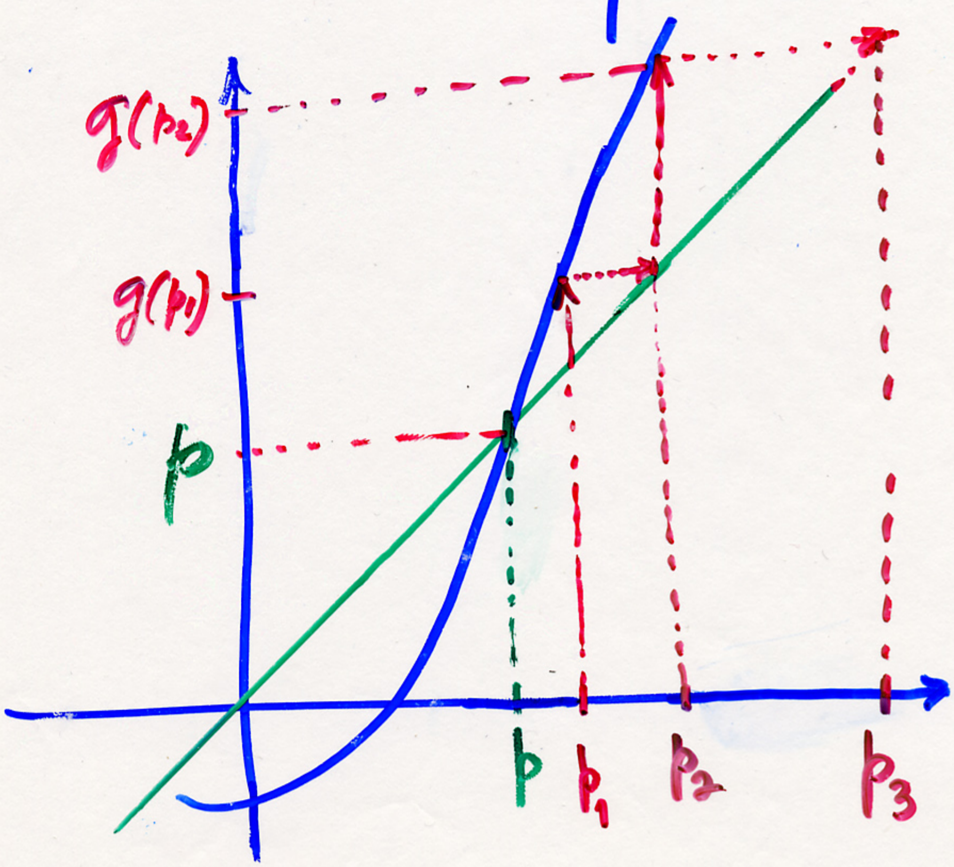
1) Choose p_0

2) $p_1 = g(p_0)$

$\vdots$

n+1) $p_n = g(p_{n-1})$


p_1
p_3
p
p_2
-1
1
$g(p_1)$
$g(p_2)$


$g(p_2)$
$g(p_1)$
p
p
p_1
p_2
p_3

Theorem :- (Fixed point)

① $g(x)$ is a Continuous function on $[a,b]$.

② The Values $g(x)$ belong to $[a,b]$. $g(x) \in [a,b]$

③ $g'(x)$ exists in (a,b)

④ there exists $0 < K < 1$, Such that

$$|g'(x)| \le K, \quad x \in (a,b)$$

then

① There is a unique fixed point $p \in [a,b]$.

② the Sequence $\{p_n\}_{n=1}^{\infty}$ obtained from the functional iteration

$$p_n = g(p_{n-1}), \quad n \ge 1,$$

where p_0 an arbitrary point in $[a,b]$, Converges to the unique point p in $[a,b]$.

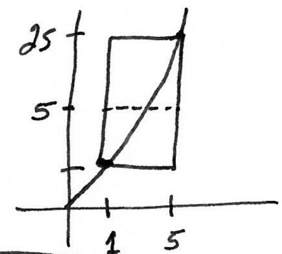
I.– EXISTENCE OF FIXED POINT

Proof.– $g: [a,b] \longrightarrow [a,b]$

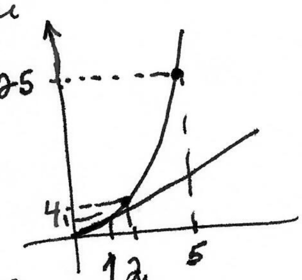
Not always true. For example,

$$g(x) = x^2$$

$$g: [2,5] \longrightarrow [2,25]$$

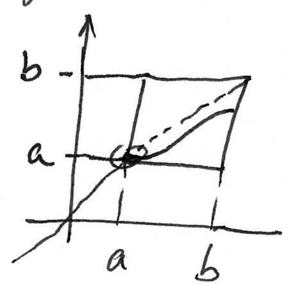


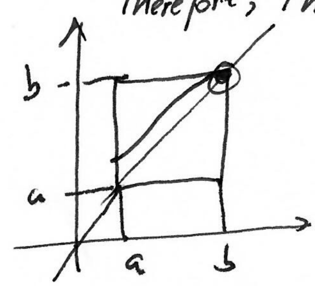
But if ② $g([a,b]) \subset [a,b]$
and if ① $g(x)$ is continuous ~~on~~ $[a,b]$, then



Two possibilities:

a) $g(a) = a$ or $g(b) = b$



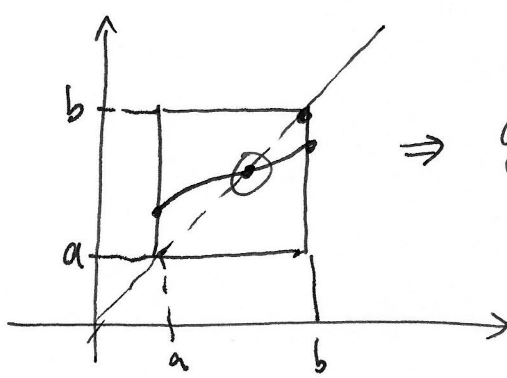


Therefore, there is at least one fixed point in $[a,b]$.

or

b) $g(a) \neq a$ and $g(b) \neq b$.

It means



$\Rightarrow$ $g(a) > a$
and
$g(b) < b$

Therefore, the function

$$f(x) = g(x) - x \quad \text{continuous on } [a,b]$$

has the property that

$$f(a) = g(a) - a > 0$$

$$f(b) = g(b) - b < 0$$

Intermediate Value Thm implies that

there exists $p \in (a,b)$ Such that

$$f(p) = 0 \iff \boxed{g(p) = p} \quad \text{fixed point for } g(x).$$

It might be more than one p.

II.- <u>UNIQUENESS OF FIXED POINT.</u>

Assume there are two fixed point $p_1 \neq q_1$, $(p<q)$
$[p,q] \subset [a,b]$

Then

$$|p-q| = |g(p) - g(q)| \overset{\substack{\text{Mean}\\ \text{Value}\\ \text{Thm}}}{=\!=} |g'(\xi)|\,|p-q|$$

where $\xi \in (p,q) \subset [a,b]$

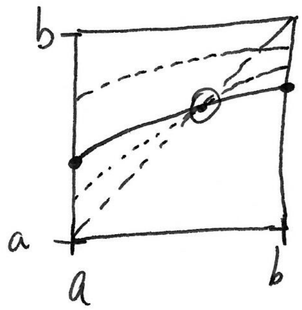
Since $|g'(x)| \leq K < 1$

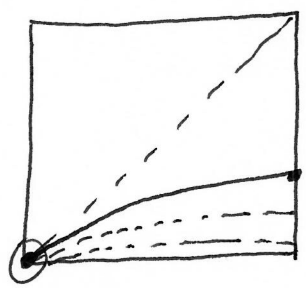
then

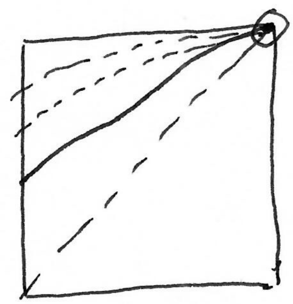
$$|p-q| = |g'(\xi)|\,|p-q| < |p-q|$$

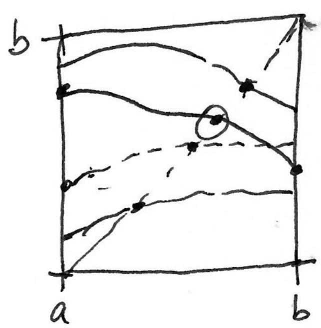
Contradiction!. Wrong assumption two fixed point $p \neq q$.

Possible situations: $|f'(x)| \leq k < 1$, for all $x \in (a,b)$

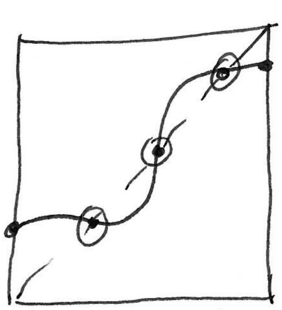




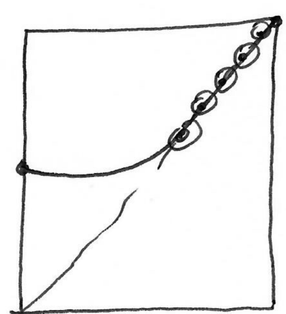




If $|g'(x)| \geq 1$ at some $x \in (a,b)$



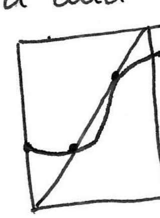
3 fixed points



Infinitely many

Can we have only two fixed points different than "a" and "b"

yes,



② Proof that the sequence $\{p_n\}_{n=1}^{\infty}$ obtained from the functional iteration

$$p_n = g(p_{n-1}) \quad n \geq 1$$

where p_0 is an arbitrary point in $[a,b]$ Converges to the unique point p in $[a,b]$.

Proof. Since g is onto $[a,b]$ then $p_n = g(p_{n-1})$ is in $[a,b]$ for all n.

and

$$|p_n - p| = |g(p_{n-1}) - \underset{\text{unique fixed point}}{g(p)}| \overset{MVT}{=} g'(\xi)\,|p_{n-1} - p|$$

for $\xi \in (a,b)$

$$\leq K\,|p_{n-1} - p|.$$

Applying this inequality inductively gives

$$|p_n - p| \leq K\,|p_{n-1} - p| \leq K^2\,|p_{n-2} - p| \leq \cdots$$
$$\leq K^n\,|p_0 - p|$$

Since $0 < K < 1$

$\lim\limits_{n\to\infty} K^n = 0$ then, $\lim\limits_{n\to\infty} |p_n - p| \leq \lim\limits_{n\to\infty} K^n\,|p_0 - p|$

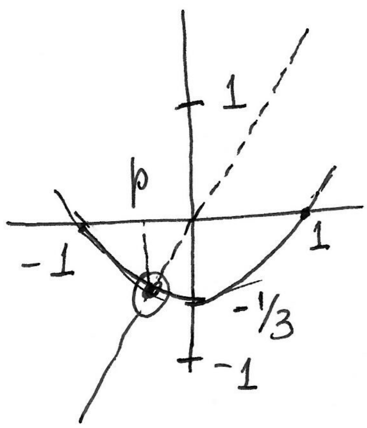
$$\leq \lim_{n\to\infty} K^n (b-a) = 0$$

or $p_n \xrightarrow[n\to\infty]{} p$

Back to our example

$$g(x) = \frac{x^2-1}{3}$$

$$g'(x) = \frac{2}{3}x.$$



Then $|g'(x)| \le \frac{2}{3}$ for $x \in (-1,1)$

Also $g(x)$ is continuous

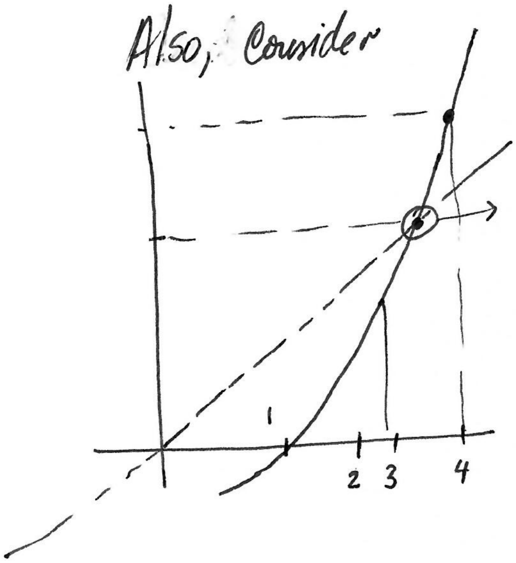
$g'(x)$ exists

and $g([-1,1]) \subset [-1,1]$.

$g(x)$ is onto $[-1,1]$.

Therefore, fixed point theorem applies and there is a unique $p \in (-1,1)$ fixed point.

Also, Consider



$p = \frac{1}{2}(3+\sqrt{13}) \simeq 3.3028$

Unique fixed point in $(3,4)$

However, $g(3) = \frac{8}{3}$, $g(4) = 5$

So $g([3,4]) \not\subset [3,4]$ <u>Not onto</u>

Also $g'(7/2) = \frac{2}{3}\frac{7}{2} = \frac{7}{3} > 1$

and functional iteration $p_n = g(p_{n-1})$ does not converge.

Corollary 2.4 If $g(x)$ satisfies the hypothesis of thm 2.3, then bounds for the error involved by approx. "p" by p_n are given by

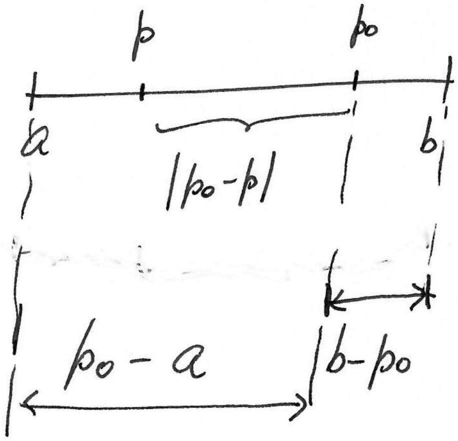
a) $$|p_n - p| \le K^n \max\{p_0 - a, b - p_0\}$$

b) $$|p_n - p| \le \frac{K^n}{1-K} |p_1 - p_0|, \quad \text{for all } n \ge 1.$$

Proof.– a) It's very easy to prove.

From previous thm

$$|p_n - p| \le K|p_{n-1} - p| \le K^2 |p_{n-2} - p| \le \cdots \le K^n |p_0 - p|$$



$$\le K^n \max\{p_0 - a, b - p_0\}.$$

If $p_0 \ge p$ and $a \le p \le b \Rightarrow -b \le -p \le -a$

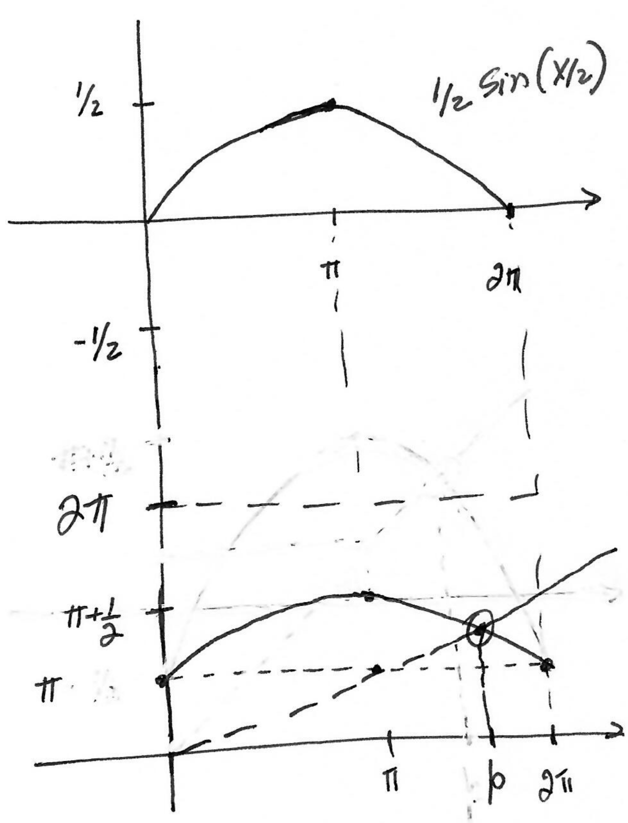
$$p_0 - b \le p_0 - p \le p_0 - a$$

If $p_0 \le p$ and $a \le p \le b \Rightarrow a - p_0 \le p - p_0 \le b - p_0$

Sect 2.2 problem #7

Thm 2.2 to show $g(x) = \pi + \frac{1}{2}\sin(x/2)$ has a unique fixed point on $[0, 2\pi]$.

Graph:



a) $g[0,2\pi] \subset [\pi, \pi+1/2] \subset [0, 2\pi] \Rightarrow g(x)$ is onto!

b) $g(x)$ is continuous and has derivatives continuous of all orders.

c) Also, $g'(x) = \frac{1}{4}\cos(x/2) \Rightarrow |g'(x)| \le \frac{1}{4}|\cos(x/2)| \le 1/4 < 1$ everywhere.

$\Rightarrow$ $g(x)$ has a unique fixed point on $[0, 2\pi]$ and it can be approximated by the functional iteration $p_n = g(p_{n-1})$.

Estimation of # of iterations needed for $|p_n - p| \leq 10^{-2}$:

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$$

In our case, $k = \frac{1}{4}$, choose $p_0 = \pi$

and $\max\{p_0 - a, b - p_0\} = \pi$

Therefore, $|p_n - p| \leq \boxed{\left(\frac{1}{4}\right)^n \pi < 10^{-2}}$

applying $\log_{10}$ $\Rightarrow$ $n \log_{10}(1/4) + \log_{10}\pi < -2 \Rightarrow -n \log_{10}(4) < -2 - \log_{10}\pi$

$$n > \frac{2 + \log_{10}\pi}{\log_{10} 4} \simeq 4.15$$

So for $n \geq 5$ the approx. will be within 10^{-2} accuracy.