0.1 Derivation of Finite Difference (FD) Approximations

0.1.1 Centered Difference for $u'(x)$

A second order finite difference approximation for $u'(x)$ at $x = \bar{x}$ is given by

$$D_0 u(\bar{x}) = \frac{1}{2h}[u(\bar{x}+h) - u(\bar{x}-h)] \tag{1}$$

with an approximation for the truncation error given by the term $E(h) \approx \frac{h^2}{6} u'''(\bar{x})$.

Proof.-
Assuming that u is 4th continuously differentiable in a neighborhood of $\bar{x}$, the FD formula (1) and its truncation error can be obtained from Taylor expansions of u at the points $x+h$ and $x-h$. In fact,

$$u(\bar{x}-h) = u(\bar{x}) - hu'(\bar{x}) + \frac{h^2}{2}u''(\bar{x}) - \frac{h^3}{3!}u'''(\bar{x}) + \frac{h^4}{4!}u^{(4)}(\bar{x}) - \frac{h^5}{5!}u^{(5)}(\beta), \tag{2}$$

$$u(\bar{x}+h) = u(\bar{x}) + hu'(\bar{x}) + \frac{h^2}{2}u''(\bar{x}) + \frac{h^3}{3!}u'''(\bar{x}) + \frac{h^4}{4!}u^{(4)}(\bar{x}) + \frac{h^5}{5!}u^{(5)}(\xi), \tag{3}$$

for $\beta \in (\bar{x}-h, \bar{x})$ and $\xi \in (\bar{x}, \bar{x}+h)$. Then, by subtracting $u(\bar{x}+h) - u(\bar{x}-h)$, we obtain

$$u(\bar{x}+h) - u(\bar{x}-h) = 2hu'(\bar{x}) + \frac{h^3}{3}u'''(\bar{x}) + \mathcal{O}(h^5) \tag{4}$$

Therefore,

$$D_0 u(\bar{x}) = \frac{1}{2h}[u(\bar{x}+h) - u(\bar{x}-h)] = u'(\bar{x}) + \frac{h^2}{6}u'''(\bar{x}) + \mathcal{O}(h^4) \tag{5}$$

0.1.2 Centered Difference for $u''(x)$

A second order finite difference approximation for $u''(x)$ at $x = \bar{x}$ is given by

$$D^2 u(\bar{x}) = \frac{1}{h^2}[u(\bar{x}+h) - 2u(\bar{x}) + u(\bar{x}-h)] \tag{6}$$

with a truncation error given by the term $E(h) = \frac{h^2}{12}u^{(4)}(\gamma)$, where $\gamma \in (\bar{x}-h, \bar{x}+h)$.

Proof.-
Assuming that u is 4th continuously differentiable in a neighborhood of $\bar{x}$, the FD formula (6) and its truncation error can be obtained by adding the above Taylor expansions (2) and (3) of u at the points $x+h$ and $x-h$. In fact,

$$u(\bar{x}+h)+u(\bar{x}-h)=2u(\bar{x})+h^2u''(\bar{x})+\frac{h^4}{4!}[u''''(\xi)+u''''(\beta)] \tag{7}$$

Therefore,

$$D^2u(\bar{x})=\frac{1}{h^2}[u(\bar{x}+h)-2u(\bar{x})+u(\bar{x}-h)]=u''(\bar{x})+\frac{h^2}{12}u''''(\gamma) \tag{8}$$

In this last step the intermediate value theorem has been used to transform the error term. In fact,

$$\frac{h^4}{4!}[u''''(\xi)+u''''(\beta)]=\frac{h^4}{12}\left[\frac{u''''(\xi)+u''''(\beta)}{2}\right]=\frac{h^4}{12}u''''(\gamma),$$

where $\gamma\in(\beta,\xi)\subset(\bar{x}-h,\bar{x}+h)$.

0.1.3 Non-Symmetric Third Order Approximation for $u'(x)$

A third order approximation D_3u for $u'(x)$ at $x=\bar{x}$ using the values of u at the neighbor points $\bar{x}-2h$, $\bar{x}-h$, $\bar{x}$, and $\bar{x}+h$, where $h>0$ is given by

$$D_3u(\bar{x})=\frac{1}{3!h}\left[u(\bar{x}-2h)-6\,u(\bar{x}-h)+3\,u(\bar{x})+2\,u(\bar{x}+h)\right] \tag{9}$$

with an approximation for the truncation error given by $E(h)\approx\frac{h^3}{12}u^{(4)}(\bar{x})$

Proof.-
The method of undetermined coefficients will be employed. This is $D_3u(\bar{x})$ will be represented as

$$D_3u(\bar{x})=c_{-2}\,u(\bar{x}-2h)+c_{-1}\,u(\bar{x}-h)+c_0\,u(\bar{x})+c_1\,u(\bar{x}+h), \tag{10}$$

and we will determine the unknown coefficients c_i $i=-2..1$ by requiring that

$$D_3u(\bar{x})=u'(\bar{x})+\mathcal{O}(h^p), \tag{11}$$

where p is the highest possible. We will assume that u is 5th continuously differentiable in a neighborhood of $\bar{x}$., then

$$u(\bar{x}-2h)=u(\bar{x})-2hu'(\bar{x})+4\frac{h^2}{2}u''(\bar{x})-8\frac{h^3}{3!}u'''(\bar{x})+16\frac{h^4}{4!}u''''(\bar{x})+\mathcal{O}(h^5) \tag{12}$$

$$u(\bar{x}-h)=u(\bar{x})-hu'(\bar{x})+\frac{h^2}{2}u''(\bar{x})-\frac{h^3}{3!}u'''(\bar{x})+\frac{h^4}{4!}u''''(\bar{x})+\mathcal{O}(h^5) \tag{13}$$

$$u(\bar{x})=u(\bar{x}) \tag{14}$$

$$u(\bar{x}+h)=u(\bar{x})+hu'(\bar{x})+\frac{h^2}{2}u''(\bar{x})+\frac{h^3}{3!}u'''(\bar{x})+\frac{h^4}{4!}u''''(\bar{x})+\mathcal{O}(h^5) \tag{15}$$

Substitution of these Taylor expansions into (10) leads to

$$
\begin{aligned}
D_3u(\bar{x}) &= (c_{-2}+c_{-1}+c_0+c_1)u(\bar{x}) + h\,(--2c_{-2}-c_{-1}+c_1)u'(\bar{x}) + \\
&\frac{h^2}{2}(4c_{-2}+c_{-1}+c_1)u''(\bar{x}) + \frac{h^3}{3!}(-8c_{-2}-c_{-1}+c_1)u'''(\bar{x}) + \\
&\frac{h^4}{4!}(16c_{-2}+c_{-1}+c_1)u''''(\bar{x}) + \mathcal{O}(h^5)
\end{aligned}
\tag{16}
$$

To get an approximation of $u'(\bar{x})$ of $\mathcal{O}(h^p)$ for the highest possible p, we impose the following 4 conditions for the coefficients:

$$u(\bar{x}):\ c_{-2}+c_{-1}+c_0+c_1 = 0 \tag{17}$$
$$u'(\bar{x}):\ h\,(-2c_{-2}-c_{-1}+c_1) = 1 \tag{18}$$
$$u''(\bar{x}):\ \frac{h^2}{2}(4c_{-2}+c_{-1}+c_1) = 0 \tag{19}$$
$$u'''(\bar{x}):\ \frac{h^3}{3!}(-8c_{-2}-c_{-1}+c_1) = 0. \tag{20}$$

$\Rightarrow D_3u(\bar{x}) = u'(x) + \frac{h^4}{4!}(16c_{-2}+c_{-1}+c_1) * u'''(\bar{x})$

Obviously, this is the best approximation we can obtain when using 4 points (4 unknowns and 4 equations). This leads to the Vandermonde system of equations with matrix

$$
\begin{pmatrix}
1 & 1 & 1 & 1 \\
-2h & -h & 0 & h \\
4\frac{h^2}{2} & \frac{h^2}{2} & 0 & \frac{h^2}{2} \\
-8\frac{h^3}{3!} & -\frac{h^3}{3!} & 0 & \frac{h^3}{3!}
\end{pmatrix}
$$

This system has a unique solution given by

$$c_{-2} = \frac{1}{3!h}, \quad c_{-1} = \frac{-6}{3!h}, \quad c_0 = \frac{3}{3!h}, \quad c_1 = \frac{2}{3!h}$$

Therefore,

$$D_3u(\bar{x}) = \frac{1}{3!h}\left[u(\bar{x}-2h) - 6\,u(\bar{x}-h) + 3\,u(\bar{x}) + 2\,u(\bar{x}+h)\right] \cancel{+E(h)}. \tag{21}$$

$u'(x) + E(h)$

The approximation for the *truncation error* $E(h)$ can be computed from the last term of (16)

$$E(h) \approx \frac{h^4}{4!}(16c_{-2}+c_{-1}+c_1)u''''(\bar{x}) = \frac{h^4}{4!}\frac{1}{6h}[16-6+2]u''''(\bar{x}) = \frac{h^3}{12}u''''(\bar{x}) \tag{22}$$

Then,

$$\frac{u(\bar{x}-2h) - 6u(\bar{x}-h) + 3u(\bar{x}) + 2u(\bar{x}+h)}{6h} = u'(x) + \frac{h^3}{12}u^4(\bar{x}) + \dots$$

or

$$u'(x) = \frac{u(\bar{x}-2h) - 6u(\bar{x}-h) + 3u(\bar{x}) + 2u(\bar{x}+h)}{6h} - \frac{h^3}{12}u^{(4)}(\bar{x}) + \mathcal{O}(h^4)$$

FINITE DIFFERENCE FORMULAS ON NON-UNIFORM GRIDS.

We want to approximate $u^{(k)}(\bar{x})$ using a formula consisting of $n>k$ points: $x_1, x_2, .., x_n$ (not necessarily uniform) and $\bar{x}$ is not a grid point x_j, $j=1,..,n$.

Problem: Determine c_j $(j=1,..,n)$ such that

$$\boxed{c_1 u(x_1) + c_2 u(x_2) + \cdots + c_n u(x_n) = u^{(k)}(\bar{x}) + \text{Highest possible Error.}} \qquad (1)$$

First, consider Taylor series expansion about $\bar{x}$ for each x_j.

$$u(x_j) = u(\bar{x}) + (x_j - \bar{x})\, u'(\bar{x}) + \cdots \frac{(x_j-\bar{x})^n}{n!} u^{(n)}(\bar{x}) + \cdots$$

Subst into (1) leads to

$$\left(\sum_{j=1}^{n} c_j\right) u(\bar{x}) + \left(\sum_{j=1}^{n} c_j (x_j - \bar{x})\right) u'(\bar{x}) + \cdots + \frac{1}{k!}\left(\sum_{j=1}^{n} c_j (x_j-\bar{x})^k\right) u^{(k)}(\bar{x}) + \cdots + \frac{1}{(n-1)!}\left(\sum_{j=1}^{n} c_j (x_j - \bar{x})^{n-1}\right) u^{(n-1)}(\bar{x}) \qquad (2)$$

$$= u^{(k)}(\bar{x}) + \text{Error.}$$

Linear system of n equs. n unknowns to be solved for the c_j, $j=1,.., n$.

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Remark:

$$\text{Error} = \overbrace{\frac{1}{n!}\left(\sum_{j=1}^{n} c_j (x_j - \bar{x})^n\right) u^{(n)}(\bar{x})}^{\text{leading order term}} + \cdots$$

In order to get the best possible approximation for $u^{(k)}(\bar{x})$ $(k<n)$, we impose the conditions:

$$\sum_{j=1}^{n} c_j = 0 \; : \quad c_1 + c_2 + \dots + c_n = 0$$

$$\sum_{j=1}^{n} c_j (x_j - \bar{x}) = 0 \; : \quad c_1(x_1-\bar{x}) + c_2(x_2-\bar{x}) + \dots + c_n(x_n-\bar{x}) = 0$$

$$\vdots$$

$$\frac{1}{k!}\sum_{j=1}^{n} c_j (x_j-\bar{x})^k = 1 \; : \quad c_1\frac{(x_1-\bar{x})^k}{k!} + c_2\frac{(x_2-\bar{x})^k}{k!} + \dots + c_n\frac{(x_n-\bar{x})^k}{k!} = 1$$

$$\vdots$$

$$\frac{1}{(n-1)!}\sum_{j=1}^{n} c_j (x_j-\bar{x})^{n-1} = 0 \; : \quad c_1\frac{(x_1-\bar{x})^{n-1}}{(n-1)!} + \dots + c_n\frac{(x_n-\bar{x})^{n-1}}{(n-1)!} = 0$$

As a result, we have n equations with n unknowns: $c_1, c_2, \dots, c_n$.

$$\begin{array}{c} u: \\ u' \\ u'' \\ \vdots \\ u^{(k)} \\ \vdots \\ u^{(n-1)} \end{array} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1-\bar{x} & x_2-\bar{x} & \cdots & x_n-\bar{x} \\ \frac{(x_1-\bar{x})^2}{2} & \frac{(x_2-\bar{x})^2}{2} & \cdots & \frac{(x_n-\bar{x})^2}{2} \\ \vdots & & & \\ \frac{(x_1-\bar{x})^k}{k!} & \frac{(x_2-\bar{x})^k}{k!} & \cdots & \frac{(x_n-\bar{x})^k}{k!} \\ \vdots & & & \\ \frac{(x_1-\bar{x})^{n-1}}{(n-1)!} & \frac{(x_2-\bar{x})^{n-1}}{(n-1)!} & \cdots & \frac{(x_n-\bar{x})^{n-1}}{(n-1)!} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{k+1} \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Once $c_1, c_2, .., c_n$ have been determined the first two Leading order error terms are given by $(k<n)$

$$\text{Error} = \left[\frac{1}{n!} \sum_{j=1}^{n} c_j (x_j-\bar{x})^n\right] u^{(n)}(\bar{x}) + \left[\frac{1}{(n+1)!} \sum_{j=1}^{n} c_j (x_j-\bar{x})^{n+1}\right] u^{(n+1)}(\bar{x})$$

$$u'(x) = \frac{u(x+h) - u(x-h)}{2h} - \frac{h^2}{6} u'''(x) + O(h^4)$$

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} - \frac{h^2}{12} u^{(4)}(x) + O(h^4)$$

$$u'(x) = \frac{u(x-2h) - 6u(x-h) + 3u(x) + 2u(x+h)}{6h} - \frac{h^3}{12} u^{(4)}(x) + O(h^4)$$

Table 3-1 Difference approximations using more than three points

Derivative	Finite-difference representation	Equation
$\left.\frac{\partial^3 u}{\partial x^3}\right)_{i,j} =$	$\frac{u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j}}{2h^3} + O(h^2)$	(3-38)
$\left.\frac{\partial^4 u}{\partial x^4}\right)_{i,j} =$	$\frac{u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{h^4} + O(h^2)$	(3-39) → obtain using fdstencil.m
$\left.\frac{\partial^2 u}{\partial x^2}\right)_{i,j} =$	$\frac{-u_{i+3,j} + 4u_{i+2,j} - 5u_{i+1,j} + 2u_{i,j}}{h^2} + O(h^2)$	(3-40)
$\left.\frac{\partial^3 u}{\partial x^3}\right)_{i,j} =$	$\frac{-3u_{i+4,j} + 14u_{i+3,j} - 24u_{i+2,j} + 18u_{i+1,j} - 5u_{i,j}}{2h^3} + O(h^2)$	(3-41)
$\left.\frac{\partial^2 u}{\partial x^2}\right)_{i,j} =$	$\frac{2u_{i,j} - 5u_{i-1,j} + 4u_{i-2,j} - u_{i-3,j}}{h^2} + O(h^2)$	(3-42)
$\left.\frac{\partial^3 u}{\partial x^3}\right)_{i,j} =$	$\frac{5u_{i,j} - 18u_{i-1,j} + 24u_{i-2,j} - 14u_{i-3,j} + 3u_{i-4,j}}{2h^3} + O(h^2)$	(3-43) Add this one. to hwk Already there
$\left.\frac{\partial u}{\partial x}\right)_{i,j} =$	$\frac{-u_{i+2,j} + 8u_{i+1,j} - 8u_{i-1,j} + u_{i-2,j}}{12h} + O(h^4)$	(3-44)
$\left.\frac{\partial^2 u}{\partial x^2}\right)_{i,j} =$	$\frac{-u_{i+2,j} + 16u_{i+1,j} - 30u_{i,j} + 16u_{i-1,j} - u_{i-2,j}}{12h^2} + O(h^4)$	(3-45) Already in Leveque hwk

Add: Find Centered of $U''(x)$ using 7 points. Already in Leveque hwk.

Table 3-2 Difference approximations for mixed partial derivatives

Derivative	Finite-difference representation	Equation
$\left.\frac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\frac{1}{\Delta x}\left(\frac{u_{i+1,j} - u_{i+1,j-1}}{\Delta y} - \frac{u_{i,j} - u_{i,j-1}}{\Delta y}\right) + O(\Delta x, \Delta y)$	(3-46)
$\left.\frac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\frac{1}{\Delta x}\left(\frac{u_{i,j+1} - u_{i,j}}{\Delta y} - \frac{u_{i-1,j+1} - u_{i-1,j}}{\Delta y}\right) + O(\Delta x, \Delta y)$	(3-47)
$\left.\frac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\frac{1}{\Delta x}\left(\frac{u_{i,j} - u_{i,j-1}}{\Delta y} - \frac{u_{i-1,j} - u_{i-1,j-1}}{\Delta y}\right) + O(\Delta x, \Delta y)$	(3-48)
$\left.\frac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\frac{1}{\Delta x}\left(\frac{u_{i+1,j+1} - u_{i+1,j}}{\Delta y} - \frac{u_{i,j+1} - u_{i,j}}{\Delta y}\right) + O(\Delta x, \Delta y)$	(3-49)
$\left.\frac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\frac{1}{\Delta x}\left(\frac{u_{i+1,j+1} - u_{i+1,j-1}}{2\,\Delta y} - \frac{u_{i,j+1} - u_{i,j-1}}{2\,\Delta y}\right) + O[\Delta x, (\Delta y)^2]$	(3-50)
$\left.\frac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\frac{1}{\Delta x}\left(\frac{u_{i,j+1} - u_{i,j-1}}{2\,\Delta y} - \frac{u_{i-1,j+1} - u_{i-1,j-1}}{2\,\Delta y}\right) + O[\Delta x, (\Delta y)^2]$	(3-51)
$\left.\frac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\frac{1}{2\,\Delta x}\left(\frac{u_{i+1,j+1} - u_{i+1,j-1}}{2\,\Delta y} - \frac{u_{i-1,j+1} - u_{i-1,j-1}}{2\,\Delta y}\right) + O[(\Delta x)^2, (\Delta y)^2]$	(3-52)
$\left.\frac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\frac{1}{2\,\Delta x}\left(\frac{u_{i+1,j+1} - u_{i+1,j}}{\Delta y} - \frac{u_{i-1,j+1} - u_{i-1,j}}{\Delta y}\right) + O[(\Delta x)^2, \Delta y]$	(3-53)
$\left.\frac{\partial^2 u}{\partial x\,\partial y}\right)_{i,j} =$	$\frac{1}{2\,\Delta x}\left(\frac{u_{i+1,j} - u_{i+1,j-1}}{\Delta y} - \frac{u_{i-1,j} - u_{i-1,j-1}}{\Delta y}\right) + O[(\Delta x)^2, \Delta y]$	(3-54)

Table 3.1. Comparison of formulae to evaluate $d\bar{T}/dx$ at $x = 1.0$ for $\boxed{\bar{T}(x) = e^x}$

Case	Algebraic formula	$\left[\frac{d\bar{T}}{dx}\right]_j$	Error	Leading term in T.E.
Exact	—	2.7183	—	—
3PT SYM	$(\bar{T}_{j+1} - \bar{T}_{j-1})/2\Delta x$	2.7228	0.4533×10^{-2}	0.4531×10^{-2}
FOR DIFF	$(\bar{T}_{j+1} - \bar{T}_j)/\Delta x$	2.8588	0.1406×10^{-0}	0.1359×10^{-0}
BACK DIFF	$(\bar{T}_j - \bar{T}_{j-1})/\Delta x$	2.5868	-0.1315×10^{-0}	-0.1359×10^{-0}
3PT ASYM	$(-1.5\bar{T}_j + 2\bar{T}_{j+1} - 0.5\bar{T}_{j+2})/\Delta x$	2.7085	-0.9773×10^{-2}	-0.9061×10^{-2}
5PT SYM	$(\bar{T}_{j-2} - 8\bar{T}_{j-1} + 8\bar{T}_{j+1} - \bar{T}_{j+2})/12\Delta x$	2.7183	-0.9072×10^{-5}	-0.9061×10^{-5}

$T(x) = e^x$

Table 3.2. Comparison of formulae to evaluate $d^2\bar{T}/dx^2$ at $x = 1.0$

Case	Algebraic formula	$\left[\frac{d\bar{T}^2}{dx^2}\right]_j$	Error	Leading term in T.E.
Exact	—	2.7183	—	—
3PT SYM	$(\bar{T}_{j-1} - 2\bar{T}_j + \bar{T}_{j+1})/\Delta x^2$	2.7205	0.2266×10^{-2}	0.2265×10^{-2}
3PT ASYM	$(\bar{T}_j - 2\bar{T}_{j+1} + \bar{T}_{j+2})/\Delta x^2$	3.0067	0.2884×10^{-0}	0.2718×10^{-0}
5PT SYM	$(-\bar{T}_{j-2} + 16\bar{T}_{j-1} - 30\bar{T}_j + 16\bar{T}_{j+1} - \bar{T}_{j+2})/12\Delta x^2$	2.7183	-0.3023×10^{-5}	-0.3020×10^{-5}

Table 3.3. Truncation error leading term (algebraic): $d\bar{T}/dx$

Case	Algebraic formula	Truncation error leading term
3PT SYM	$(\bar{T}_{j+1} - \bar{T}_{j-1})/2\Delta x$	$\Delta x^2 \bar{T}_{xxx}/6$
FOR DIFF	$(\bar{T}_{j+1} - \bar{T}_j)/\Delta x$	$\Delta x \bar{T}_{xx}/2$
BACK DIFF	$(\bar{T}_j - \bar{T}_{j-1})/\Delta x$	$-\Delta x \bar{T}_{xx}/2$
3PT ASYM	$(-1.5\bar{T}_j + 2\bar{T}_{j+1} - 0.5\bar{T}_{j+2})/\Delta x$	$-\Delta x^2 \bar{T}_{xxx}/3$
5PT SYM	$(\bar{T}_{j-2} - 8\bar{T}_{j-1} + 8\bar{T}_{j+1} - \bar{T}_{j+2})/12\Delta x$	$-\Delta x^4 \bar{T}_{xxxxx}/30$

good Hwk

Table 3.4. Truncation error leading term (algebraic): $d^2\bar{T}/dx^2$

Case	Algebraic formula	Truncation error leading term
3PT SYM	$(\bar{T}_{j-1} - 2\bar{T}_j + \bar{T}_{j+1})/\Delta x^2$	$\Delta x^2 \bar{T}_{xxxx}/12$
3PT ASYM	$(\bar{T}_j - 2\bar{T}_{j+1} + \bar{T}_{j+2})/\Delta x^2$	$\Delta x \bar{T}_{xxx}$
5PT SYM	$(-\bar{T}_{j-2} + 16\bar{T}_{j-1} - 30\bar{T}_j + 16\bar{T}_{j+1} - \bar{T}_{j+2})/12\Delta x^2$	$-\Delta x^4 \bar{T}_{xxxxxx}/90$

good Hwk. It's already a hwk prob. Exercise 1.2.

Hornbeck (1975)

	f_{i-2}	f_{i-1}	f_i	f_{i+1}	f_{i+2}	
$2hf'(x_i) =$		−1	0	1		2 more than the order
$h^2f''(x_i) =$		1	−2	1		1 more
$2h^3f'''(x_i) =$	−1	2	0	−2	1	2 more
$h^4f^{iv}(x_i) =$	1	−4	6	−4	1	1 more

$+ \; O(h)^2$

(*a*) Representations of $O(h)^2$

	f_{i-3}	f_{i-2}	f_{i-1}	f_i	f_{i+1}	f_{i+2}	f_{i+3}	
$12hf'(x_i) =$		1	−8	0	8	−1		4 more
$12h^2f''(x_i) =$		−1	16	−30	16	−1		3 more
$8h^3f'''(x_i) =$	1	−8	13	0	−13	8	−1	4 more
$6h^4f^{iv}(x_i) =$	−1	12	−39	56	−39	12	−1	3 more

$+ \; O(h)^4$ than the order

(*b*) Representations of $O(h)^4$

Fig. 3.4 Central difference representations.

Summarizing

If n: # points
k: order of deriv.
p: order of truncation error.
and centered FD is employed
Then, a) If k is odd $n-k=p$
b) If k is even $p = n-k+1$

	f_i	f_{i+1}	f_{i+2}	f_{i+3}	f_{i+4}	f_{i+5}
$2hf'(x_i) =$	−3	4	−1			
$h^2f''(x_i) =$	2	−5	4	−1		
$2h^3f'''(x_i) =$	−5	18	−24	14	−3	
$h^4f^{iv}(x_i) =$	3	−14	26	−24	11	−2

$+ \; O(h)^2$

(*a*) Forward difference representations

	f_{i-5}	f_{i-4}	f_{i-3}	f_{i-2}	f_{i-1}	f_i
$2hf'(x_i) =$				1	−4	3
$h^2f''(x_i) =$			−1	4	−5	2
$2h^3f'''(x_i) =$		3	−14	24	−18	5
$h^4f^{iv}(x_i) =$	−2	11	−24	26	−14	3

$+ \; O(h)^2$

(*b*) Backward difference representations

Fig. 3.3 Forward and backward difference representations of $O(h)^2$.

If
n: # points
k: order of derivative
p: Order of Truncation error.
2 more everyone!
And forward, backward, or simply non-symm FD is employed
2 more Then,

$$\boxed{n-k=p}$$

The relationship is:
points in stencil − Order of derivative $= O(r) =$ Order of truncation error.
For centered FD $O(r)$ is increase by one due to cancellations.