# Eingenvalues and Eigenvectors.

Consider the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

If we multiply $A$ by $\vec{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

we obtain

$$A\vec{v} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad A\vec{y} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

So $\quad A\vec{v} = \vec{v} \quad , \quad A\vec{y} = \vec{y}$

However, $A \neq I$. If we pick another vector, for example $\vec{z} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, then

$$A\vec{z} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \end{bmatrix}$$

and for $\vec{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$A\vec{w} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

Also, for $\vec{x} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$

$$A\vec{x} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$$

Therefore, $A\vec{w} = 4\vec{w}$ and $A\vec{x} = 4\vec{x}$.

Obviously, the vectors $\vec{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are special vectors. But not only these two vectors have special properties when $A$ is multiplied by them. <u>Every multiple of them has the same property</u>. In fact

$$A(\alpha\vec{v}) \overset{\text{matrix arithmetic}}{=} \alpha A\vec{v} = \alpha\vec{v}, \quad \text{for any } \alpha \text{ real.}$$

$$A(\alpha\vec{w}) = \alpha A\vec{w} = \alpha 4\vec{w} = 4(\alpha\vec{w}), \quad \text{for any } \alpha$$

Vectors as $\vec{v}$ and $\vec{w}$ receive the special name of <u>eigenvectors of the matrix A.</u> Also, the scalar numbers $\lambda_1 = 1$, $\lambda_2 = 4$ receive the special name of <u>eigenvalues of the matrix A.</u>

<u>Question</u>: Given a matrix $A_{(n\times n)}$, How can we find its eigenvalues and eigenvectors?

We are looking for scalars $\lambda$ and vectors $\vec{v}$ that satisfy the equation

$$\boxed{A\vec{x} = \lambda\vec{x}} \qquad (9.1)$$

This equation is equivalent to

$$A\vec{x} - \lambda\vec{x} = 0 \quad \text{or} \quad A\vec{x} - \lambda I\vec{x} = 0$$

$$\text{or} \quad \boxed{(A - \lambda I)\vec{x} = \vec{0}} \quad \text{or} \quad \boxed{B\vec{x} = \vec{0}} \qquad (9.2)$$

where <u>$B = A - \lambda I$</u>

(9.2) is a homogeneous linear system of equations

Obviously, $\vec{x} = \vec{0}$ is a solution of (9.1) for any $\lambda$

We are interested in finding non-trivial Solutions of

$$A\hat{x} = \lambda \hat{x} \qquad (9.1)$$

or

$$\left.\begin{aligned} (A-\lambda I)\,\vec{x} &= \vec{0} \\ B\,\hat{x} &= \vec{0} \end{aligned}\right\} \begin{array}{l} (9.2) \\ B = A-\lambda I \end{array}$$

What condition should we impose on $B$ to guarantee the existence of non-trivial solutions?

$$\det(B) = \det(A-\lambda I) = |A-\lambda I| = 0. \qquad (10.1)$$

This condition reduces to an equation for $\lambda$. By finding the solutions of (10.1), the eigenvalues are determined. Once an eigenvalue has been determined. It can be substituted in (9.2) to find its non-trivial solutions or eigenvectors.

Back to our example to illustrate the procedure.

If $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$, then

$$|A - \lambda I| = \left| \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix}$$

$$= (3-\lambda)(2-\lambda) - 2 =$$
$$= (\lambda-3)(\lambda-2) - 2 = \lambda^2 - 5\lambda + 6 - 2$$
$$= \lambda^2 - 5\lambda + 4 = (\lambda-1)(\lambda-4)$$

Therefore, $|A - \lambda I| = 0$ if $\lambda_1 = 1$, $\lambda_2 = 4$.

Now, we will find the eigenvectors associated to each one of the eigenvalues.

For $\lambda_1 = 1$, we have to solve the homogeneous system $(A - 1.I)\vec{x} = \vec{0}$ (11.1)

We know, this system has non-trivial solutions Why?

or

$$\begin{bmatrix} 3-1 & 1 \\ 2 & 2-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{cases} 2v_1 + v_2 = 0 \\ 2v_1 + v_2 = 0 \end{cases}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} 2v_1 + v_2 = 0 \\ v_2 = -2v_1 \end{array}$$

or $\vec{v} = t\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} t \\ -2t \end{pmatrix}$ for any $t \neq 0$.

This is a family of eigenvectors associated to the eigenvalue $\lambda_1 = 1$.

We will do the same work to find the eigenvectors for $\lambda_2 = 4$

$$\begin{bmatrix} 3-4 & 1 \\ 2 & 2-4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

row-reducing

$$\begin{cases} -w_1 + w_2 = 0 \\ 2w_1 - 2w_2 = 0 \end{cases}$$

$$\begin{pmatrix} -1 & 1 & 0 \\ 2 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow -w_1 + w_2 = 0 \Rightarrow w_2 = w_1$ or $\vec{w} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, for any $t \neq 0$.

or $\vec{w} = \begin{pmatrix} t \\ t \end{pmatrix}$