

4.4 Composite Numerical Integration

COMPOSITE TRAPEZOIDAL.

Thm. - ① $f \in C^2[a, b]$

② $P = \{a = x_0 < x_1 < \dots < x_n = b\}$ uniform $\left(\begin{array}{l} = x_i - x_{i-1} \\ i=1, \dots, n \end{array} \right)$
partition of $[a, b]$

Then,

$$\int_a^b f(x) dx = \frac{h}{2} \left[f(a) + 2 \sum_{i=1}^{n-1} f(x_i) + f(b) \right] - \underbrace{\frac{b-a}{12} h^2 f''(\mu)}_{\text{error term}}, \quad \mu \in (a, b).$$

Proof. - For a single subinterval $[x_i, x_{i+1}]$, $i=0, 1, \dots, n-1$

$$\int_{x_i}^{x_{i+1}} f(x) dx = \frac{h}{2} [f(x_i) + f(x_{i+1})] - \frac{h^3}{12} f''(\xi_i) \quad \xi_i \in (x_i, x_{i+1})$$

Therefore,

$$\int_a^b f(x) dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x) dx$$

$$\begin{aligned}
&= \frac{h}{2} \sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})] - \frac{h^3}{12} \sum_{i=0}^{n-1} f''(\xi_i) = \\
&= \frac{h}{2} \left[\overset{f(a)}{f(x_0)} + f(x_1) + f(x_1) + f(x_2) + \dots + f(x_{n-1}) + f(x_{n-1}) + \overset{f(b)}{f(x_n)} \right] \\
&\quad - \frac{h^3}{12} \sum_{i=0}^{n-1} f''(\xi_i) \\
&= \frac{h}{2} \left[f(a) + 2 \sum_{i=1}^{n-1} f(x_i) + f(b) \right] - \frac{h^3}{12} \sum_{i=0}^{n-1} f''(\xi_i)
\end{aligned}$$

Error Expression: Since $f \in C^2[a, b]$, then

$$\min_{x \in [a, b]} (f''(x)) \leq f''(\xi_i) \leq \max_{x \in [a, b]} (f''(x))$$

$$\Rightarrow n (\min(f''(x))) \leq \sum_{i=0}^{n-1} f''(\xi_i) \leq n (\max(f''(x)))$$

Now, $h = \frac{b-a}{n} \Rightarrow n = \frac{b-a}{h}$ AVG of f''

thus, $\min(f''(x)) \leq \left(\frac{1}{n} \sum_{i=0}^{n-1} f''(\xi_i) \right) = \left(\frac{h}{b-a} \sum_{i=0}^{n-1} f''(\xi_i) \right) \leq \max(f''(x))$

IVT. implies there exists $\mu \in (a, b)$ such that

$$f''(\mu) = \frac{h}{b-a} \sum_{i=0}^{n-1} f''(\xi_i).$$

$$\Rightarrow \sum_{i=0}^{n-1} f''(\xi_i) = \frac{b-a}{h} f''(\mu).$$

$$\Rightarrow -\frac{h^3}{12} \sum_{i=0}^{n-1} f''(\xi_i) = -\frac{b-a}{h} \frac{h^3}{12} f''(\mu)$$

or Error = $-\frac{(b-a)}{12} h^2 f''(\mu)$, $\mu \in [a, b]$.
 $\hookrightarrow \underline{O(h^2)}$.

Example 3. - (bar).

$$\int_0^1 e^{-x^2} dx$$

How many points should be used to compute above integral with an error of at most $\frac{1}{2} \times 10^{-4}$.

Using the error form.

$$\left| -\frac{(b-a)}{12} \left(\frac{b-a}{n}\right)^2 f''(\mu) \right| \leq \frac{1}{2} \times 10^{-4}$$

$$\frac{1}{12} \frac{1}{n^2} |f''(\mu)| \leq \frac{1}{2} \times 10^{-4} \Rightarrow n^2 \geq \frac{2}{12} |f''(\mu)| \times 10^4$$

$$f'(x) = -2x e^{-x^2}, \quad f''(x) = e^{-x^2} (-2 + 4x^2)$$

$$\Rightarrow |f''(x)| \leq e^{-x^2} |4x^2 - 2| \leq 1.2 = 2$$

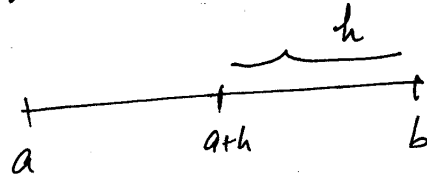
$$\therefore n^2 \geq \frac{1}{3} \times 10^4 \Rightarrow \boxed{n \geq 58}$$

Composite Simpson.

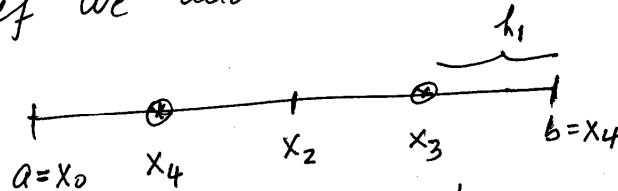
Consider first Single Simpson rule on $[a, b]$.

$$\int_a^b f(x) dx = \frac{h}{3} [f(a) + 4f(a+h) + f(b)]$$

$$h = \frac{b-a}{2}$$



Now, if we add two more points



$$h_1 = \frac{b-a}{4}$$

$$\int_a^b f(x) dx = \int_{x_0=a}^{x_2} f(x) dx + \int_{x_2}^{x_4=b} f(x) dx =$$

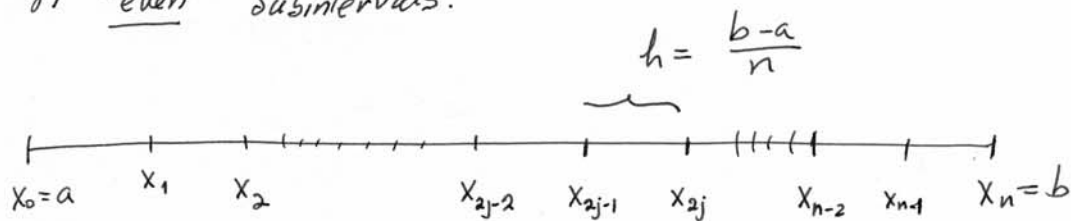
$$= \frac{h_1}{3} [f(x_0) + 4f(x_1) + f(x_2)] + \frac{h_1}{3} [f(x_2) + 4f(x_3) + f(x_4)] + \epsilon_{\text{error}}$$

$$= \frac{h_1}{3} [f(x_0) + 2f(x_2) + 4(f(x_1) + f(x_3)) + f(x_4)] +$$

$$+ \left(-\frac{h_1}{90} f^{(4)}(\xi_1) - \frac{h_1}{90} f^{(4)}(\xi_2) \right)$$

$$x_0 < \xi_1 < x_2, \quad x_2 < \xi_2 < x_4.$$

In general, subdividing the interval $[a, b]$ into n "even" subintervals.



$$\int_a^b f(x) dx = \int_{x_0}^{x_2} + \int_{x_2}^{x_4} + \dots + \int_{x_{2j-2}}^{x_{2j}} + \dots + \int_{x_{n-2}}^{x_n} = \sum_{j=1}^{n/2} \int_{x_{2j-2}}^{x_{2j}} f(x) dx$$

$$= \sum_{j=1}^{n/2} \left\{ \frac{h}{3} [f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})] - \frac{h^5}{90} f^{(4)}(\xi_j) \right\}$$

$$= \frac{h}{3} \left[f(x_0) + 2 \sum_{j=1}^{n/2-1} f(x_{2j}) + 4 \sum_{j=1}^{n/2} f(x_{2j-1}) + f(x_n) \right] - \frac{h^5}{90} \sum_{j=1}^{n/2} f^{(4)}(\xi_j)$$

$\xi_j \in (x_{2j-2}, x_{2j})$

It can be proved that

$$-\frac{h^5}{90} \sum_{j=1}^{n/2} f^{(4)}(\xi_j) = -\frac{b-a}{180} h^4 f^{(4)}(\mu)$$

→ follow proof for trapezoidal rule.

Summarizing,

Thm. - $f \in C^4[a, b]$, n even, $h = \frac{b-a}{n}$, $x_j = a + j \cdot h$
 $j = 0, 1, \dots, n$.

Then,

$$\int_a^b f(x) dx = \frac{h}{3} \left[f(a) + 2 \sum_{j=1}^{n/2-1} f(x_{2j}) + 4 \sum_{j=1}^{n/2} f(x_{2j-1}) + f(b) \right]$$

$$- \frac{b-a}{80} h^4 f^{(4)}(\mu)$$

$\mu \in (a, b)$.

Simpson's Rule

Simpson's Rule:

$$\int_{x_0}^{x_2} f(x) dx = \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(\xi).$$

Since the error term involves the fourth derivative of f , Simpson's rule gives exact results when applied to any polynomial of degree three or less.

EXAMPLE 1

The Trapezoidal rule for a function f on the interval $[0, 2]$ is

$$\int_0^2 f(x) dx \approx f(0) + f(2),$$

and Simpson's rule for f on $[0, 2]$ is

$$\int_0^2 f(x) dx \approx \frac{1}{3} [f(0) + 4f(1) + f(2)].$$

The results to three places for some elementary functions are summarized in Table 4.7. Notice that in each instance Simpson's Rule is significantly better. ■

Table 4.7

$f(x)$	x^2	x^4	$1/(x+1)$	$\sqrt{1+x^2}$	$\sin x$	e^x
Exact value	2.667	6.400	1.099	2.958	1.416	6.389
Trapezoidal	4.000	16.000	1.333	3.326	0.909	8.389
Simpson's	2.667	6.667	1.111	2.964	1.425	6.421