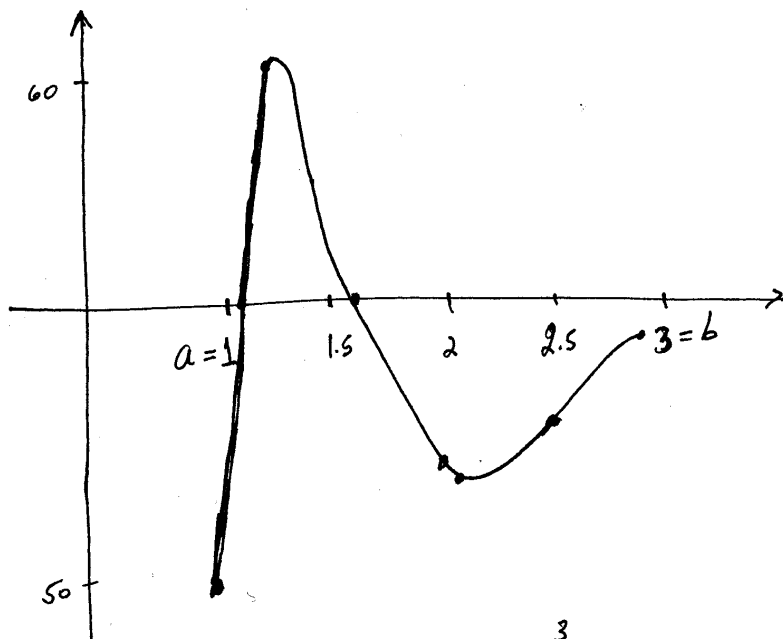


An Adaptive Simpson's Scheme.

Consider a function like

$$f(x) = \frac{100}{x^2} \sin\left(\frac{10}{x}\right)$$



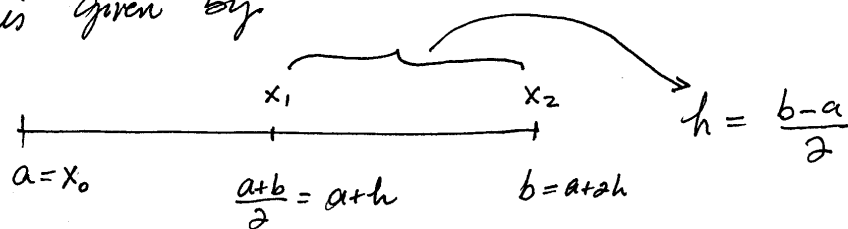
We want to approximate: $\int_1^3 f(x) dx$.

Obviously in the subinterval $[1, 2]$ the function changes more rapidly than over the subinterval $[2, 3]$.

We would like to use a Simpson rule (composite) to approximate this integral. However due to the behavior of $f(x)$, we would like to use a non-uniform partition of $[1, 3]$ with more points in $[1, 2]$ than in $[2, 3]$.

AS₂

In general, Simpson over an interval $[a, b]$ for $f(x)$ is given by



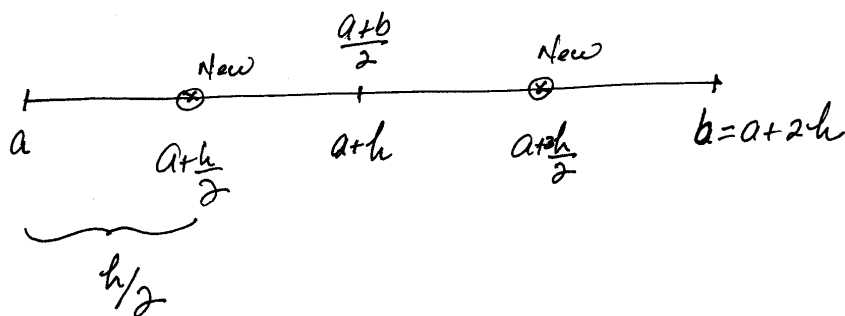
$$\int_a^b f(x) dx = \frac{h}{3} [f(a) + f(a+h) + f(b)] - \frac{h^5}{90} f^{(4)}(\xi)$$

or

$a < \xi < b$

$$I = \int_a^b f(x) dx = S(a, b) - \frac{h^5}{90} f^{(4)}(\xi)$$

Adding two more points and keeping a uniform partition.



AS₂'

We can approximate $f(x)$ with composite

Simpson for $n=4$ (5 points)

of Subintervals.

$$\begin{aligned} \int_a^b f(x) dx &= \int_a^{a+h} f(x) dx + \int_{a+h}^b f(x) dx = \\ &= \frac{h/2}{3} \left[f(a) + 4f\left(a + \frac{h}{2}\right) + f(a+h) \right] + \\ &+ \frac{h/2}{3} \left[f(a+h) + 4f\left(a + 3\frac{h}{2}\right) + f(b) \right] - \frac{b-a}{180} \left(\frac{h}{2}\right)^4 f^{(4)}(\bar{\mu}) \\ &\quad \bar{\mu} \in (a, b). \end{aligned}$$

$$\begin{aligned} &= \frac{h}{6} \left[f(a) + 4f\left(a + \frac{h}{2}\right) + f(a+h) \right] + \\ &+ \frac{h}{6} \left[f(a+h) + 4f\left(a + 3\frac{h}{2}\right) + f(b) \right] - \frac{1}{90} \frac{b-a}{2} \frac{h^4}{16} f^{(4)}(\bar{\mu}) \\ &\quad \rightarrow = \frac{1}{16} \frac{h^5}{90} f^{(4)}(\bar{\mu}) \end{aligned}$$

New notation:

$$\int_a^b f(x) dx = S(a, a+h) + S(a+h, b) - \frac{1}{16} \frac{h^5}{90} f^{(4)}(\bar{\mu})$$

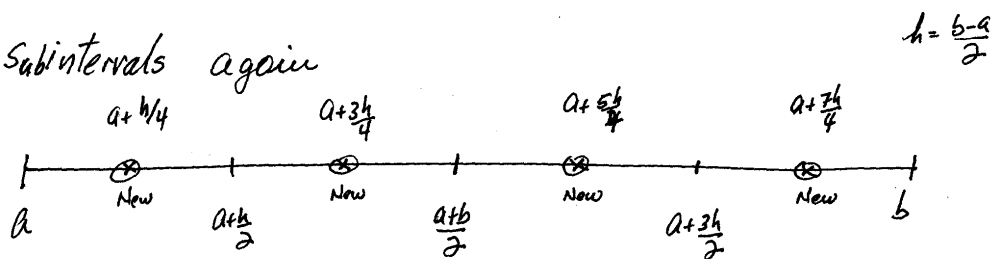
$$\therefore I = S(a, \overset{\rightarrow=a+h}{\frac{a+b}{2}}) + S(\overset{\rightarrow=a+h}{\frac{a+b}{2}}, b) - \frac{1}{16} \frac{h^5}{90} f^{(4)}(\bar{\mu}) \quad (3.1)$$

Now, let's fix certain tolerance ϵ . We want our approximation to be within this tolerance. If somehow we could evaluate the error, then it would be easy to check if the following condition is satisfied.

$$\left| \int_a^b f(x) dx - S(a, \overset{\rightarrow=a+h}{\frac{a+b}{2}}) - S(\overset{\rightarrow=a+h}{\frac{a+b}{2}}, b) \right| = |\text{Error}| < \epsilon \quad (3.2)$$

If (3.2) were not true, we subdivide the

subintervals again



Now, we approximate I by Composite Simpson with $n=8$.

$$\left| \int_a^b f(x) dx - S(a, a+\frac{h}{2}) - S(a+\frac{h}{2}, \overset{\rightarrow=a+h}{\frac{a+b}{2}}) - S(\overset{\rightarrow=a+h}{\frac{a+b}{2}}, a+\frac{3h}{2}) - S(a+\frac{3h}{2}, b) \right| \quad (3.3)$$

If we substitute

AS₄

$$\int_a^b f(x) dx \text{ by } \int_a^{\frac{a+b}{2}} f(x) dx + \int_{\frac{a+b}{2}}^b f(x) dx \text{ in (3.3)}$$

and applied the triangular inequality

$$|(3.3)| \leq \left| \int_a^{\frac{a+b}{2}} f(x) dx - S(a, a+\frac{h}{2}) - S(a+\frac{h}{2}, \frac{a+b}{2}) \right| \rightarrow (4.1)$$

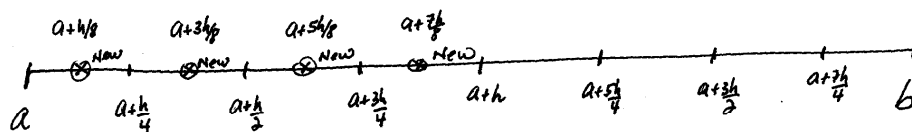
$$+ \left| \int_{\frac{a+b}{2}}^b f(x) dx - S(\frac{a+b}{2}, a+\frac{3h}{2}) - S(a+\frac{3h}{2}, b) \right| \rightarrow (4.2)$$

therefore, if $(4.1) < \frac{\varepsilon}{2}$ and $(4.2) < \frac{\varepsilon}{2}$,

$\Rightarrow |(3.3)| < \varepsilon$. and we are done.

But, if only one of them is less than $\varepsilon/2$, for example

$(4.2) < \varepsilon/2$ but $(4.1) > \varepsilon/2$. We subdivide ~~again~~ the subinterval $[a, \frac{a+b}{2}]$ but not the subinterval $[\frac{a+b}{2}, b]$.



And then we evaluate the error

$$E = \left| \int_a^{\frac{a+b}{2}} f(x) dx - S(a, a+\frac{h}{4}) - S(a+\frac{h}{4}, a+\frac{h}{2}) - S(a+\frac{h}{2}, a+\frac{3h}{4}) - S(a+\frac{3h}{4}, a+h) \right| \quad (5.1)$$

$$\text{Now, } E \leq \left| \int_a^{a+\frac{h}{2}} f(x) dx - S(a, a+\frac{h}{4}) - S(a+\frac{h}{4}, a+\frac{h}{2}) \right| \rightarrow (5.2)$$

$$+ \left| \int_{a+\frac{h}{2}}^{a+h} f(x) dx - S(a+\frac{h}{2}, a+\frac{3h}{4}) - S(a+\frac{3h}{4}, a+h) \right| \rightarrow (5.3)$$

therefore, if $(5.2) < \epsilon/4$, $(5.3) < \epsilon/4 \Rightarrow (5.1) < \epsilon/2$

and we are done.

$$\int_a^b f(x) dx \approx S(a, a+\frac{h}{4}) + S(a+\frac{h}{4}, a+\frac{h}{2}) + S(a+\frac{h}{2}, a+\frac{3h}{4}) + S(a+\frac{3h}{4}, a+h) + S(a+h, a+\frac{3h}{2}) + S(a+\frac{3h}{2}, b)$$

within an ϵ error.

Otherwise, if for example $(5.2) > \epsilon/4$, but $(5.3) < \epsilon/4$

We subdivide the subinterval $[a, a+\frac{h}{2}]$ but not the subinterval $[a+\frac{h}{2}, a+h]$ and repeat the previous procedure for $\epsilon/8$.

The previous algorithm could be implemented if we could find an efficient way to obtain a bound for

$$\left| \int_a^b f(x) dx - S(a, \frac{a+b}{2}) - S(\frac{a+b}{2}, b) \right| \quad (5.1.1)$$

for any interval $[a, b]$

We will prove next that

$$\left| \int_a^b f(x) dx - S(a, \frac{a+b}{2}) - S(\frac{a+b}{2}, b) \right| \approx \frac{1}{15} \left| S(a, b) - S(a, \frac{a+b}{2}) - S(\frac{a+b}{2}, b) \right|$$

for any interval $[a, b]$. (5.1.2).

Proof. -

AS₆

From (3.1)

$$\left| \int_a^b f(x) dx - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right) \right| = \frac{1}{16} \frac{h^5}{90} f^{(4)}(\bar{\mu}) \quad (6.1)$$

On the other hand,

$$I = S(a, b) - \frac{h^5}{90} f^{(4)}(\xi) = S\left(a, \frac{a+b}{2}\right) + S\left(\frac{a+b}{2}, b\right) - \frac{1}{16} \frac{h^5}{90} f^{(4)}(\bar{\mu})$$

$$\Rightarrow S(a, b) - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right) = \frac{h^5}{90} f^{(4)}(\xi) - \frac{1}{16} \frac{h^5}{90} f^{(4)}(\bar{\mu}) \quad (6.2)$$

if $\left[\xi = \bar{\mu} \text{ or } f^{(4)}(\xi) \approx f^{(4)}(\bar{\mu}) \right]$ Key assumption.

then the right hand side in (6.2) can be simplified as

$$\frac{15}{16} f^{(4)}(\bar{\mu}) \frac{h^5}{90}$$

$$\Rightarrow \left| \int_a^b f(x) dx - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right) \right| = \frac{1}{15} \left| S(a, b) - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right) \right| \quad (6.3)$$

Therefore, if

$$\left| S(a, b) - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right) \right| < 15\varepsilon$$

$$\Rightarrow \left| \int_a^b f(x) dx - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right) \right| < \varepsilon$$

and this is true for any interval $[a, b]$.