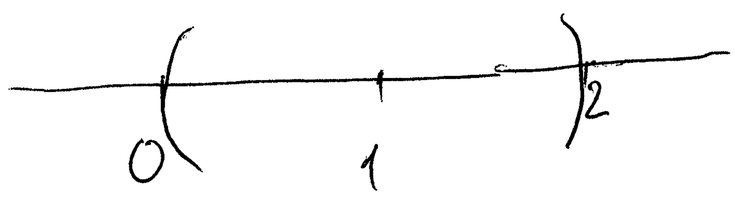
# Define logarithm of complex variables by power series

Here we would like to define logarithm of complex variables by power series, just as we did to exponential function. The familiar logarithm of real variable has the power series centered at 1 :

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n$$

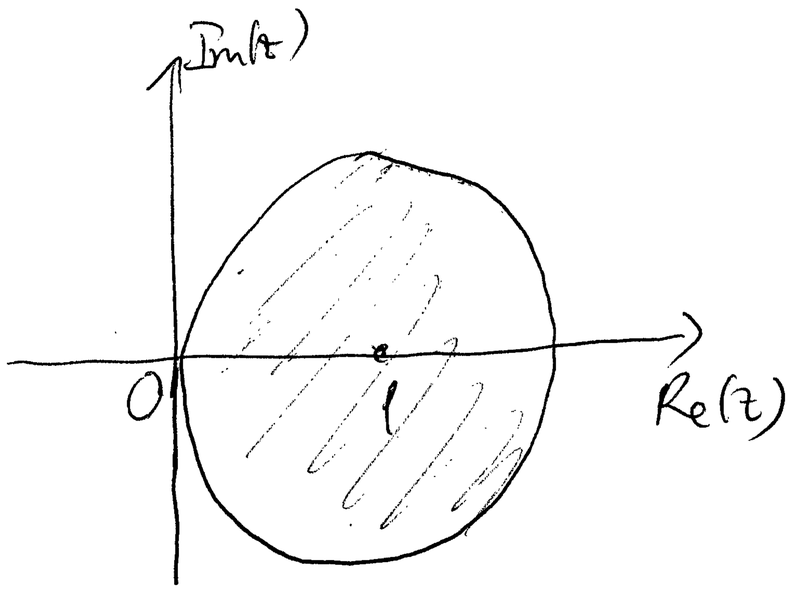



The radius of convergence is $R = 1$

Below are the steps we will follow to extend this function (analytically) to the complex domain :

1) Define series

$$f_1(z) = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^n$$




Radius of convergence is $R = 1$

Some properties of $f_1$ are ;

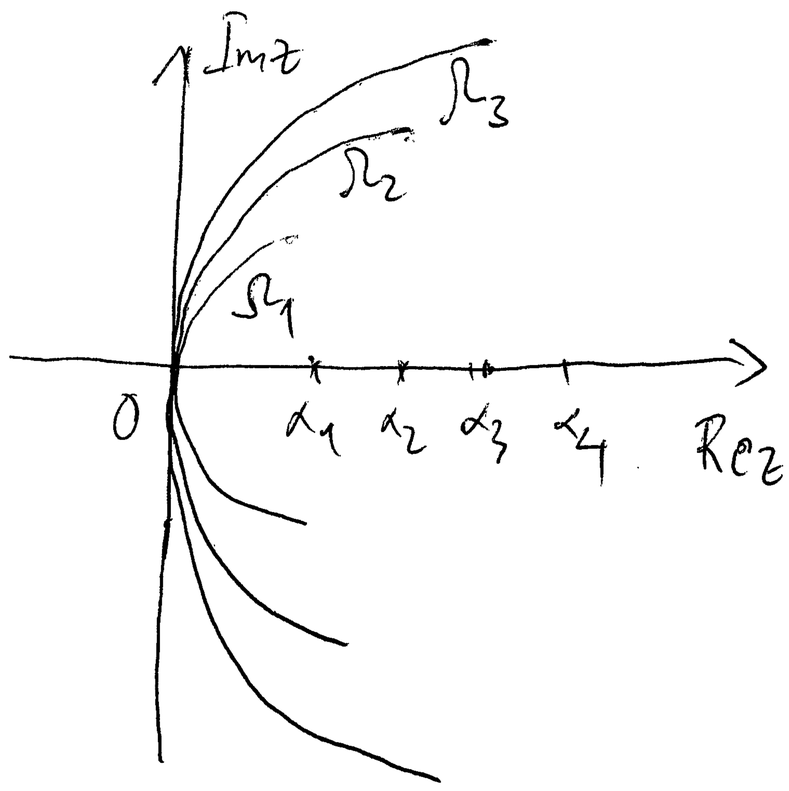
(i) $f_1$ is analytic in $\Omega_1 = \{z : |z-1| < 1\}$

(ii) $f_1'(z) = \frac{1}{z}$

(iii) If $u, v \in \Omega_1$ and $uv \in \Omega_1$ then

$$f_1(uv) = f_1(u) + f_1(v)$$

2) Extend $f_1$ analytically to $\Omega^+ = \{z : \operatorname{Re} z > 0\}$ :




For each $n \in \mathbb{N}$, we put $\alpha_n = \frac{n+1}{2}$ and

$$\Omega_n = \{z : |z - \alpha_n| < \alpha_n\}$$

We define by induction $f_n : \Omega_n^+ \to \mathbb{C}$ analytic

and

(i) $f_n|_{\Omega_{n-1}} = f_{n-1}$

(ii) If $u, v \in \Omega_n$ and $uv \in \Omega$ then $f_n(uv) = f_n(u) f_n(v)$

(iii) $f_n'(z) = \frac{1}{z}$

Then we put $\Omega^+ = \bigcup_{n=1}^{\infty} \Omega_n = \{z : \operatorname{Re} z > 0\}$ and define

$$f^+ : \Omega^+ \to \mathbb{C}$$
$$z \mapsto f_n(z) \text{ where } n \in \mathbb{N} \text{ is such that } z \in \Omega_n$$

Then we claim.

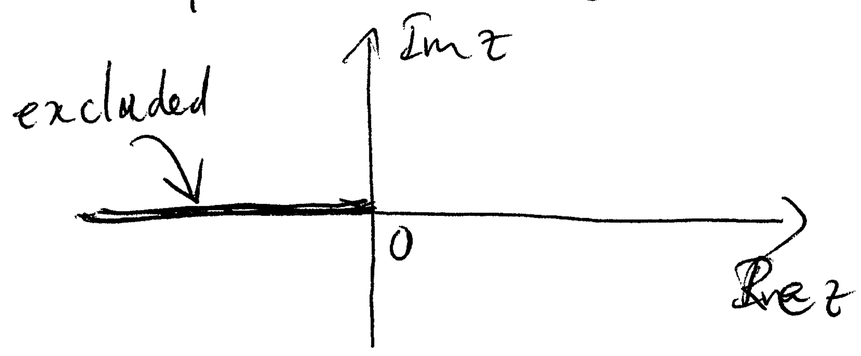
(iv) $f^+$ is analytic

(v) $f^{+\prime}(z) = \frac{1}{z}$

(vi) If $u, v \in \Omega^+$ and $uv \in \Omega^+$ then $f^+(u) + f^+(v) = f^+(uv)$

(vii) $\operatorname{Re} f^+(z) = \log|z|$

3) Extend $f^+$ analytically to $\Omega = \mathbb{C} \setminus \{z : \operatorname{Im} z = 0, \operatorname{Re} z \leq 0\}$ by the following claims :



(i) Each $z \in \Omega$ is a product of some $z_1, z_2 \in \Omega^+$: $z = z_1 z_2$

(ii) If $z_1 z_2 = z_1' z_2'$ for $z_1, z_2, z_1', z_2' \in \Omega^+$ then $f^+(z_1) + f^+(z_2) = f^+(z_1') + f^+(z_2')$.

(iii) Then we can define $f : \Omega \to \mathbb{C}$ such that $f(z) := f^+(z_1) + f^+(z_2)$, where $z = z_1 z_2$ and $z_i \in \Omega^+$

(iv) $f|_{\Omega^+} = f^+$

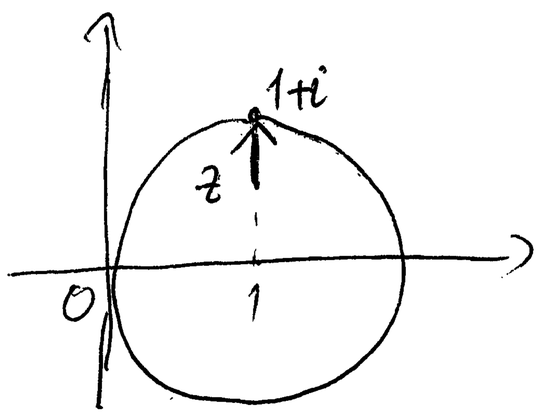
(v) $f$ analytic and $f'(z) = \frac{1}{z}$

(vi) $f$ is the unique analytic extension of $f_1$ on $\Omega$.

4) We will show that $\Omega$ is the <u>maximal</u> domain of analytic extension of $f_1$ by the following substeps

(i) Use Abel's Limit Theorem (Ahlfors, p. 41) to show that the series

$$f_1(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^n$$ converges as $z$ approaches $1+i$ vertically ~~Thus~~ to the limit




$$f(1+i) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} i^n$$

$$= \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \cdots\right) + i\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right)$$

Similarly, the above series converges as $z$ approaches $1-i$ vertically to the limit

~~Thus~~, $$f(1-i) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (-i)^n = \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \cdots\right) - i\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right)$$

We define $$\frac{\pi}{4} := 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

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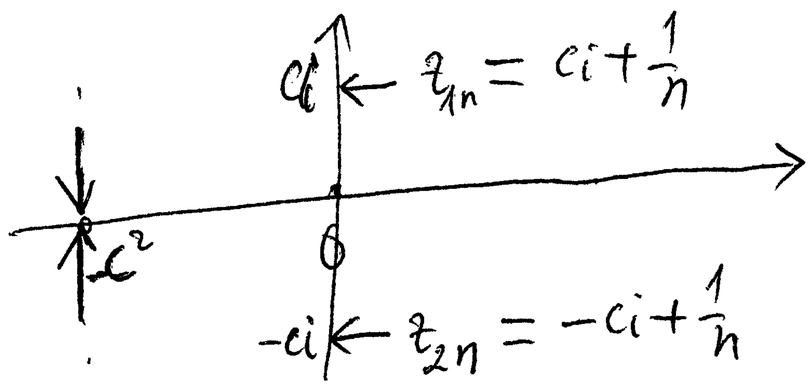
(ii) We can compute $f(i) = i\frac{\pi}{2}$ and $f(-i) = -i\frac{\pi}{2}$.

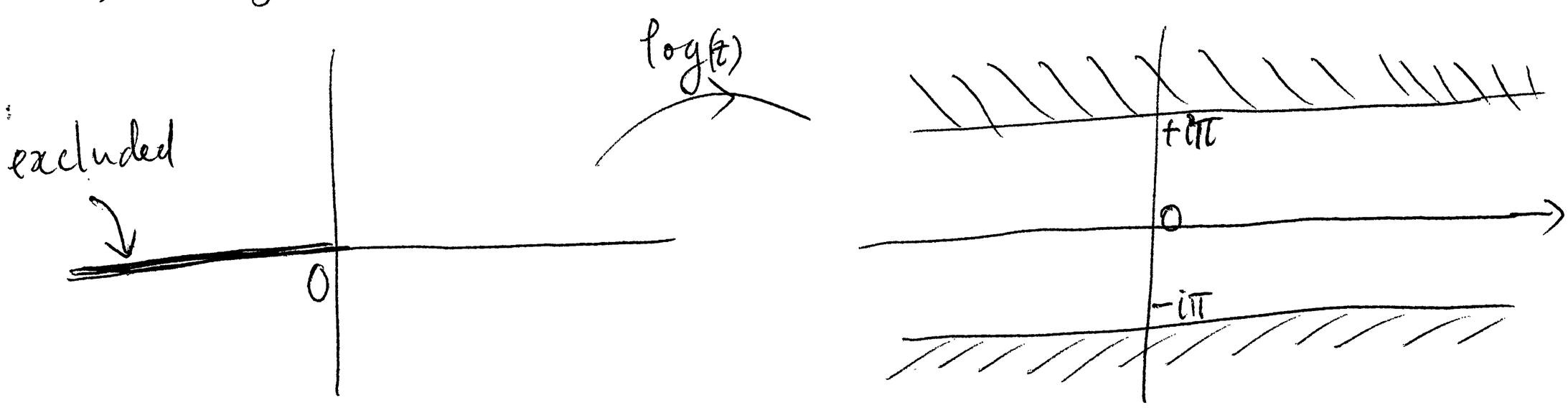
(iii) Let $c>0$. We show that the limits of $f(z)$ as $z$ approaches $-c^2$ from above and from below are different.




5) We show that $f$ is injective from $\Omega \to \mathbb{C}$

6) We show that the range of $f$ is $\{w : -\pi < Im(w) < \pi\}$

7) We conclude that function $f : \mathbb{C} \setminus \{z : Im\, z = 0, Re\, z \leq 0\} \to \{w : -\pi < Im\, w < \pi\}$ is a bijection and denoted by $\log(z)$.




The inverse function of $\log(z)$ is $g(z)$ with $g'(z) = g(z)$ and $g(0) = 1$, which will be denoted by $\exp(z)$.

<u>Remark</u>: By starting from logarithm, we not only introduce the notion of $\pi$ but also give an explicit expression for $\pi$. By starting from exponential like in AhlF we only prove the existence of $\pi$ and don't know how to calculate it.

# Details of proofs

**Step 1:** (i) $f_1$ is analytic in $\Omega_1$ because the radius of convergence is $R = 1$ and the power series is developed at center $z = 1$.

(ii) Take the derivative termwise of the series, we get

$$f_1'(z) = \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n\right)' = \sum_{n=1}^{\infty}(-1)^{n+1}(z-1)^{n-1} = \sum_{m=0}^{\infty}(1-z)^m = \frac{1}{1-(1-z)} = \frac{1}{z}$$

(iii) Let $u, v \in \Omega_1$ such that $w = uv \in \Omega_1$. Claim (haven't proved yet!) that $\Omega_1 \cap \frac{w}{\Omega_1}$ is a connected set. Put $h(z) = f_1(z) + f_1\left(\frac{w}{z}\right)$, $\forall z \in \left(\Omega_1 \cap \frac{w}{\Omega_1}\right)$

Then $h'(z) = f_1'(z) - \frac{w}{z^2} f_1'\left(\frac{w}{z}\right) = \frac{1}{z} - \frac{w}{z^2}\frac{z}{w} = 0$. thus $h \equiv \text{const}$ and particularly $h(u) = h(1)$, or $f_1(u) + f_1(v) = f_1(1) + f_1(w) = f_1(w)$.

**Step 2:**

For each $z \in \Omega_n$, we have $|z - \alpha_n| < \alpha_n$. Thus

$$\left|\frac{z}{\alpha_n} - 1\right| < 1$$

We define $f_n(z) := f_1\left(\frac{z}{\alpha_n}\right) + \log \alpha_n$

usual logarithm

(i) If $z \in \Omega_{n-1}$, we have $z \in \Omega_n$ and

$$f_n(z) = f_1\left(\frac{z}{\alpha_n}\right) + \log \alpha_n = f_1\left(\frac{z}{\alpha_{n-1}} \frac{\alpha_{n-1}}{\alpha_n}\right) + \log \alpha_n$$

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$$= f_1\left(\frac{z}{\alpha_{n-1}}\right) + \underbrace{f_1\left(\frac{\alpha_{n-1}}{\alpha_n}\right)}_{\log \frac{\alpha_{n-1}}{\alpha_n}} + \log \alpha_n \quad = f_1\left(\frac{z}{\alpha_{n-1}}\right) + \log \alpha_{n-1} = f_{n-1}(z).$$

(ii) Let $u, v \in \Omega_n$ and $uv \in \Omega_n$. We have

$$f_m(u) = f_1\left(\frac{u}{\alpha_m}\right) + \log \alpha_m$$

$$f_m(v) = f_1\left(\frac{v}{\alpha_m}\right) + \log \alpha_m \qquad \text{for every } m \geq n$$

due to the fact that $\Omega_n \subset \Omega_m$ $\forall m \geq n$. Since $\alpha_m \to \infty$, we can choose $m$ large enough such that $\frac{u}{\alpha_m}\frac{v}{\alpha_m} \in \Omega_1$ ~~Then~~ and $\frac{uv}{\alpha_m} \in \Omega_1$. Then

$$f_m(u) + f_m(v) = \left(f_1\left(\frac{u}{\alpha_m}\right) + f_1\left(\frac{v}{\alpha_m}\right)\right) + 2\log \alpha_m = f_1\left(\frac{uv}{\alpha_m^2}\right) + 2\log \alpha_m$$

$$= \left(f_1\left(\frac{uv}{\alpha_m}\frac{1}{\alpha_m}\right) + \log \alpha_m\right) + \log \alpha_m = f_m\left(\frac{uv}{\alpha_m}\right) + \log \alpha_m$$

$$= f_m(uv)$$

Because of (i), $f_m(u) = f_n(u)$, $f_m(v) = f_n(v)$, $f_m(uv) = f_n(uv)$. We then obtain

$f_n(u) + f_n(v) = f_n(uv)$.

(iii) We have $f_n'(z) = \left(f_1\left(\frac{z}{\alpha_n}\right) + \log z_n\right)' = \left(f_1\left(\frac{z}{\alpha_n}\right)\right)' = \frac{1}{\alpha_n} f_1'\left(\frac{z}{\alpha_n}\right) = \frac{1}{\alpha_n}\frac{\alpha_n}{z} = \frac{1}{z}$

(iv) Let $z_0 \in \Omega^+$. Since $\Omega^+ = \bigcup_{n=1}^{\infty} \Omega_n$, there exists $n \in \mathbb{N}$ such that $z_0 \in \Omega_n$. Thus there exists a neighborhood of $z_0$ in $\Omega^+$ that is contained in $\Omega_n$. In $\Omega_n$, $f^+$ is simply $f_n$, which is analytic thus $f^+$ is differentiable at $z_0$.

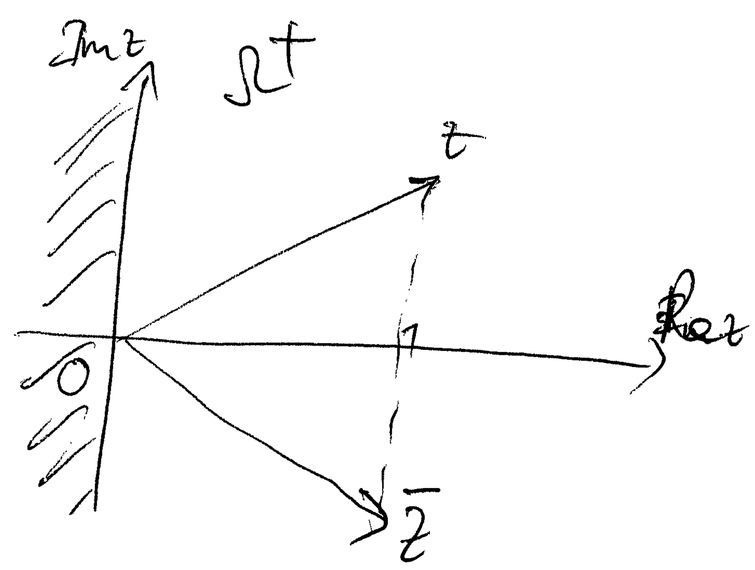
(v) Let $z_0 \in \Omega^+$. Since $z_0$ lies in some $\Omega_n$, we have

$$f^{+\prime}(z_0) = f_n'(z_0) = \frac{1}{z_0}$$

(vi) Suppose that $u \in \Omega_{n_1}$, $v \in \Omega_{n_2}$, $uv \in \Omega_{n_3}$. Take $m = \max\{n_1, n_2, n_3\}$

Then $u, v, uv \in \Omega_m$ and $f^+(u) + f^+(v) = f_m(u) + f_m(v) = f_m(uv) = f^+(uv)$.

(vii)




For each $z \in \Omega^+$, we also have $\bar{z} \in \Omega^+$.

~~Thus we~~ Moreove $z\bar{z} = |z|^2 \in \Omega^+$. Thus we can apply the identity

$$f^+(z) + f^+(\bar{z}) = f^+(z\bar{z}) = f^+(|z|^2) = \log|z|$$

Let $m \in \mathbb{N}$ be such that $z, \bar{z} \in \Omega_m$. We have

$$f^+(z) = f_m(z) = f_1\left(\frac{z}{\alpha_m}\right) + \log \alpha_m$$

$$f^+(\bar{z}) = f_m(\bar{z}) = f_1\left(\frac{\bar{z}}{\alpha_m}\right) + \log \alpha_m = \overline{f_1\left(\frac{z}{\alpha_m}\right)} + \log \alpha_m$$

(since the coefficients in the power series of $f_1$ are real)

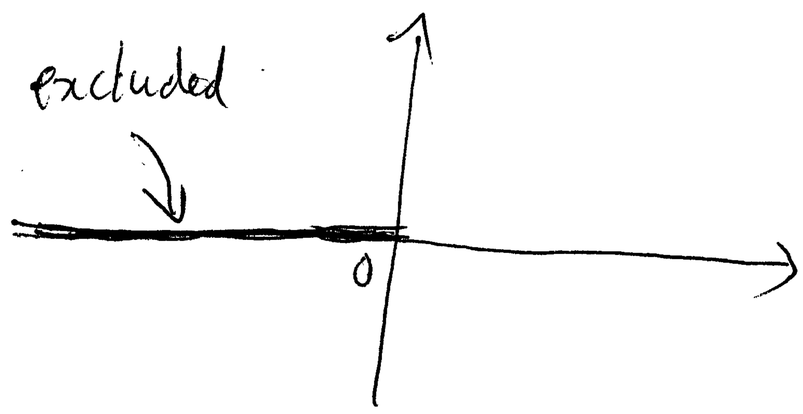
Thus

$$f^+(z) + f^+(\bar{z}) = 2\mathrm{Re}\, f_1\left(\frac{z}{\alpha_m}\right) + 2\log \alpha_m$$

$$= 2\mathrm{Re}\, f_m(z) = 2\mathrm{Re}\, f^+(z)$$

Thus $\mathrm{Re}\, f^+(z) = \log|z|$.

Step 3




$\Omega^+ = \{z : \operatorname{Re} z > 0\}$

$\Omega = \mathbb{C} \setminus \{z : \operatorname{Im} z = 0, \operatorname{Re} z \leq 0\}$

(i) Let $z = x + iy \in \Omega$. We'll find $z_1, z_2 \in \Omega^+$ such that $z = z_1 z_2$.

Put $z_1 = \delta + bi$, $z_2 = \delta + di$ where $\delta, b, d \in \mathbb{R}$; $\delta > 0$ to be chosen.

$$z = z_1 z_2 \iff \begin{cases} x = \delta^2 - bd \\ y = \delta(b+d) \end{cases} \iff \begin{cases} bd = \delta^2 - x \\ b + d = \frac{y}{\delta} \end{cases}$$

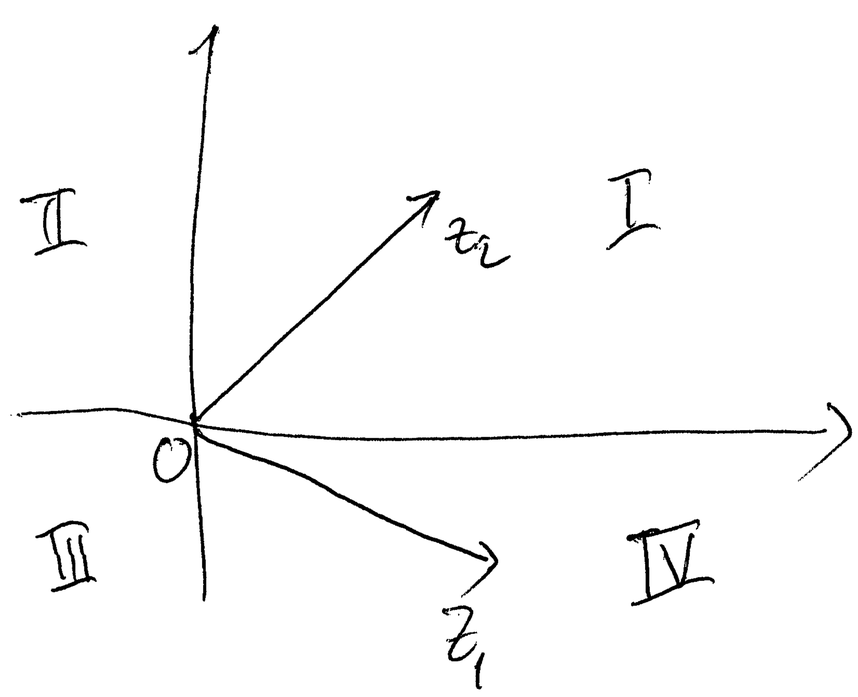
The system has solution $(b, d)$ if and only if $\left(\frac{y}{\delta}\right)^2 \geq 4(\delta^2 - x)$, which is equivalent to $y^2 \geq 4\delta^2(\delta^2 - x)$. $(*)$

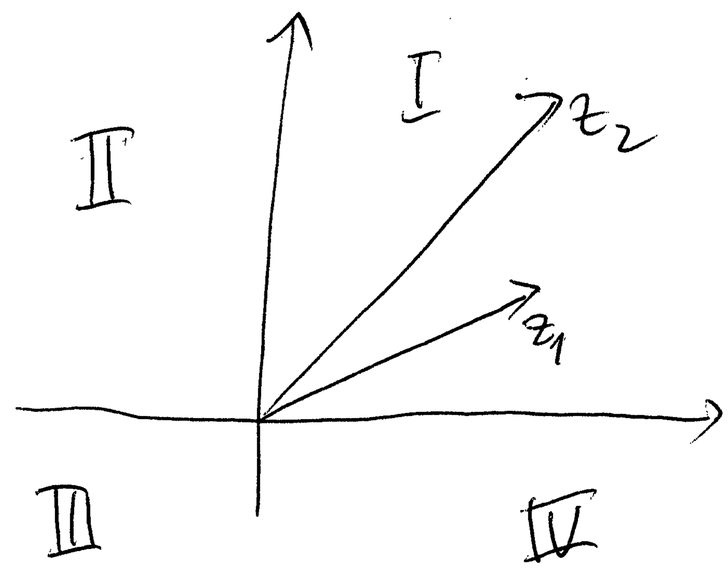
If $y = 0$, then $x > 0$ because $z \in \Omega$. Then $(*)$ is satisfied if $\delta$ is small enough.

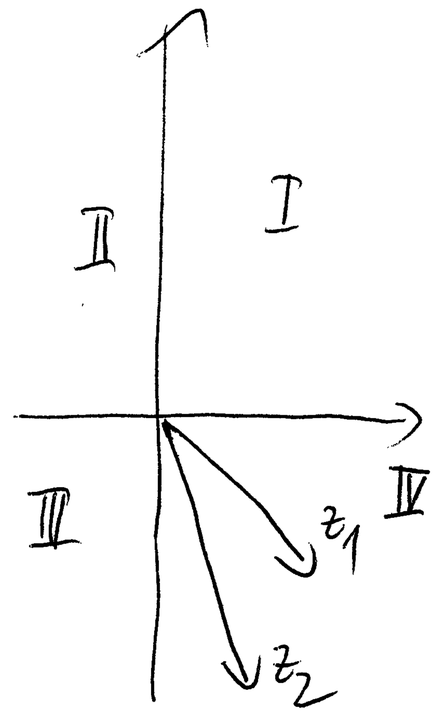
If $y \neq 0$, then $(*)$ is satisfied if $\delta$ is small enough.

(ii) Let $z_1, z_2, z_1', z_2' \in \Omega^+$ such that $z_1 z_2 = z_1' z_2' = z \in \Omega$. We'll show that

$f^+(z_1) + f^+(z_2) = f^+(z_1') + f^+(z_2')$.



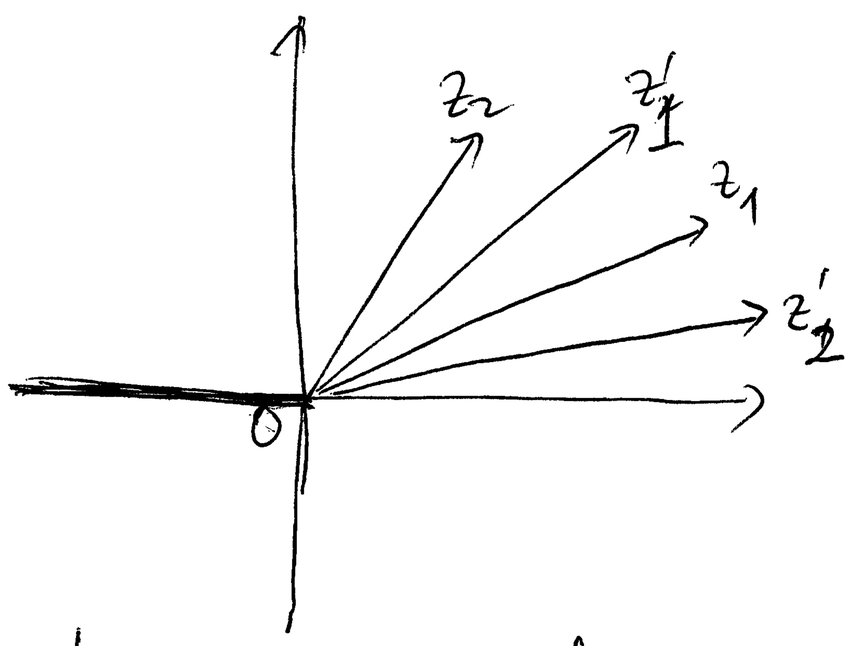
We consider 2 cases:

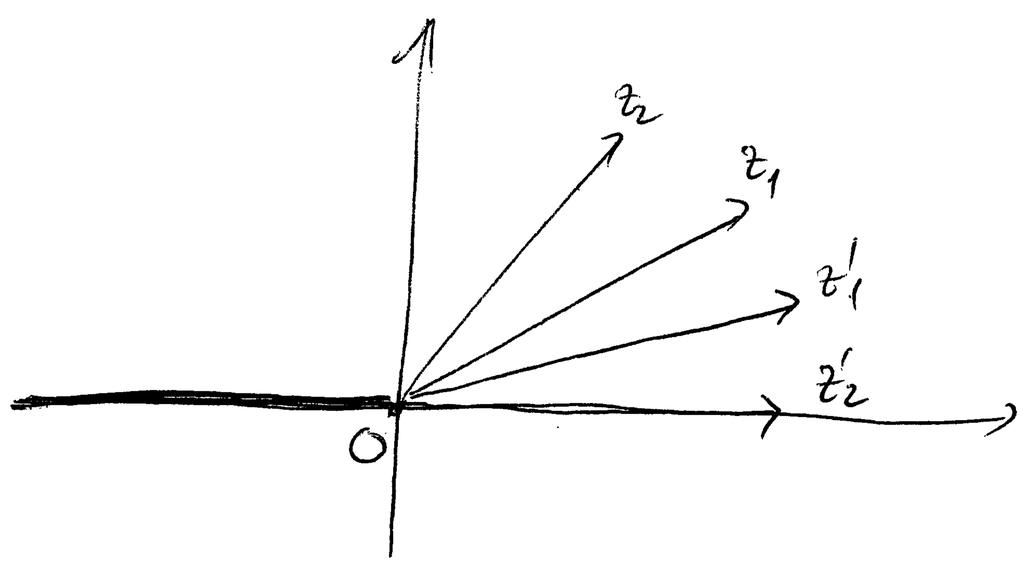
1) $z_1$ and $z_2$ belong to different quadrants.

WLOG, we can assume $z_1 \in \mathrm{IV}$ and $z_2 \in \mathrm{I}$. Then their product $z = z_1 z_2$ also belongs to either IV or I, which means $z \in \Omega^+$. Thus by (vi) in Step 2, we have $f^+(z_1) + f^+(z_2) = f^+(z_1 z_2) = f^+(z) = f^+(z_1' z_2') = f^+(z_1) + f^+(z_2)$. $(*)$

2) $z_1$ and $z_2$ belong to the same quadrant.

WLOG, we can assume $z_1, z_2 \in \mathrm{I}$ Then $z = z_1 z_2$ must be in quadrant I or II. If $z \in \mathrm{I}$ then again we have the identity $(*)$. We only consider the case $z \in \mathrm{II}$ We see that if $z_1'$ and $z_2'$ belong to different quadrants, $z$ will be in either I or IV, which is a contradiction. If $z_1'$ and $z_2'$ belong to quadrant IV, then $z = z_1' z_2'$ belong to either III or IV, which is also a contradiction. The only case left is that $z_1', z_2' \in \mathrm{I}$.




Then we have either $\operatorname{Re} \frac{z_1}{z_1'} > 0$ or $\operatorname{Re} \frac{z_1}{z_2'} > 0$ (we are using picture to give an intuitive proof. This is not rigorous!)

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WLOG, suppose that $Re\left(\frac{z_1}{z_1'}\right) > 0$. Then $\frac{z_1}{z_1'} = \frac{z_2'}{z_2} \in \Omega^+$. Thus

$$f^+(z_1) = f^+\left(\frac{z_1}{z_1'} z_1'\right) = f^+\left(\frac{z_1}{z_1'}\right) + f^+(z_1') = f^+\left(\frac{z_2'}{z_2}\right) + f^+(z_1')$$

$$= f^+(z_2') - f^+(z_2) + f^+(z_1')$$

Therefore $f^+(z_1) + f^+(z_2) = f^+(z_1') + f^+(z_2')$.

(iii) By (i) and (ii), we can define a function $f : \Omega \to \mathbb{C}$ such that $f(z) := f^+(z_1) + f^+(z_2)$ for $z = z_1 z_2$, where $z_1$ and $z_2$ are some elements in $\Omega^+$.

(iv) For each $z \in \Omega^+$, we have $z = z \cdot 1$ and $1 \in \Omega^+$. By the definition of $f$, we have $f(z) = f^+(z) + f^+(1) = f^+(z)$

(v) Take $z_0 \in \Omega$, we'll show that $f$ is analytic at $z_0$. There are $z_1, z_2 \in \Omega^+$ such that $z_0 = z_1 z_2$. We see that $\frac{z_0}{z_1} = z_2 \in \Omega^+$

Since $\Omega^+$ is an open set in $\mathbb{C}$ and the map $z \mapsto \frac{z}{z_1}$ is continuous, there exists a neighborhood $V$ of $z_0$ in $\Omega$ such that $\frac{z}{z_1} \in \Omega^+$ $\forall z \in V$.

We have $f(z) = f^+(z_1) + f^+\left(\frac{z}{z_1}\right)$ $\forall z \in V$.

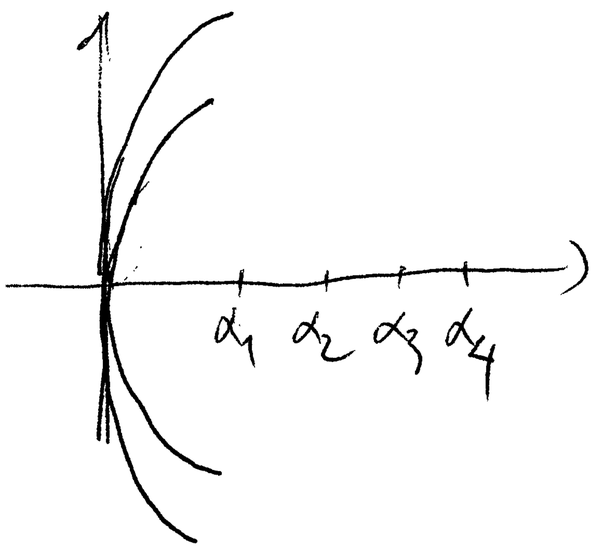
Thus $f'(z_0) = \frac{1}{z_1} f^{+\prime}\left(\frac{z_0}{z_1}\right) = \frac{1}{z_1} \frac{z_1}{z_0} = \frac{1}{z_0}$

Therefor $f$ is analytic at $z_0$ and $f'(z_0) = \frac{1}{z_0}$.

(vi) Let $g$ be another analytic extension of $f_1$ on $\Omega$. Put $g : \Omega \to \mathbb{C}$

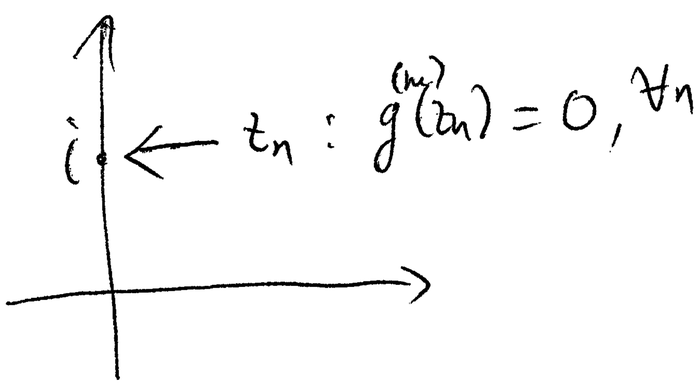
given by $g(z) = f(z) - \tilde{f}(z)$. Then $g$ is analytic on $\Omega$ and $g(z) = 0 \ \forall z \in$

Recall our notation in Step 1: $\begin{cases} \alpha_n = \frac{n+1}{2} \\ \Omega_n = \{z : |z - \alpha_n| < \alpha_n\} \end{cases}$




We see that $\alpha_2 \in \Omega_1$ and hence $g^{(m)}(\alpha_2) = 0 \ \forall m \in \mathbb{N}$

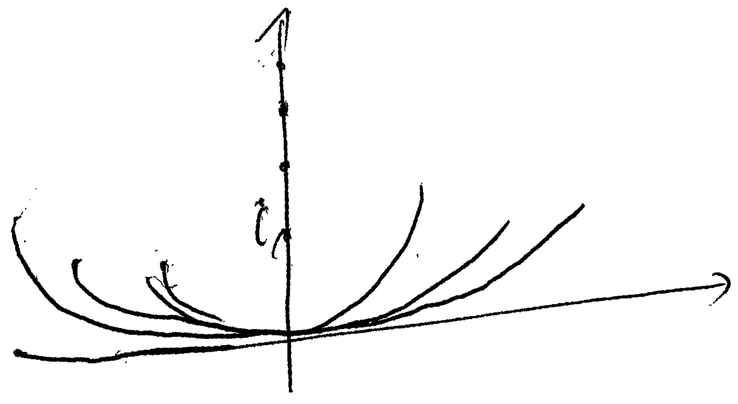
Thus, by using Taylor expansion of $g$ at $\alpha_2$, we have $g(z) = 0 \ \forall z \in \Omega_2$. Again $\alpha_3 \in \Omega_2$, and hence $g^{(m)}(\alpha_3) = 0 \ \forall m \geq 0$. By using Taylor expansion of $g$ at $\alpha_3$, we get $g(z) = 0 \ \forall z \in \Omega_3$. By continuing doing so, we get $g(z) = 0 \ \forall z \in \Omega_n$ for every $n \in \mathbb{N}$, and hence $g(z) = 0 \ \forall z \in \Omega$




Take an arbitrary sequence $(z_n)$ in $\Omega$ that approach $i$. Since $g^{(m)}$ is continuous on $\Omega$, we have $g^{(m)}(i) = \lim_{n \to \infty} g^{(m)}(z_n) = 0$.

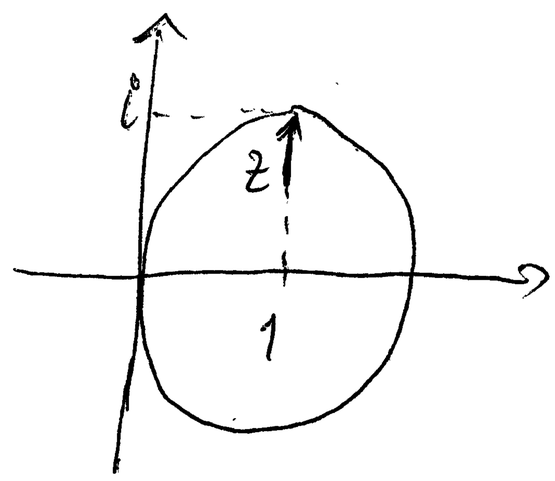
By using Taylor expansion at $i$, and then at $\frac{3}{2}i = \alpha_2 i$, and then at $\frac{5}{2}i = \alpha_3 i$, ... we get $g(z) = 0 \ \forall z : \operatorname{Im} z > 0$.

Similarly, by replacing $i$ by $-i$, we get $g(z) = 0 \ \forall z : \operatorname{Im} z < 0$.



Therefore $g(z) = 0 \ \forall z \in \Omega$, which means $f(z) = \tilde{f}(z) \ \forall z \in \Omega$.

Step 4 (i) We will show that $f_1(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^n$




converges to $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} i^n$ as $z$ approaches $1+i$ vertically up. We change the variable

$$u = \frac{z-1}{i}$$

Put $a_n = \frac{(-1)^{n+1}}{n} i^n$ We have $f_1(z) = \tilde{f}_1(u) = \sum_{n=1}^{\infty} a_n u^n$

As $z$ approaches $1+i$ vertically up, $u$ goes to $1$ from the left. Thus the quantity $\frac{|1-u|}{1-|u|} = \frac{1-u}{1-u} = 1$ remains bounded. By Abel's limit theorem (Ahlfors, p. 41), we will have $\tilde{f}_1(z) \to \tilde{f}_1(1)$ if we can show that $\sum_{n=1}^{\infty} a_n$ converges.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} i^n = \underbrace{\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \dots\right)}_{\text{Converges because we can group the } 2k-1\text{'th and } 2k\text{'th terms}} + i \underbrace{\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)}_{\text{Converges for the same reason.}}$$

We define $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

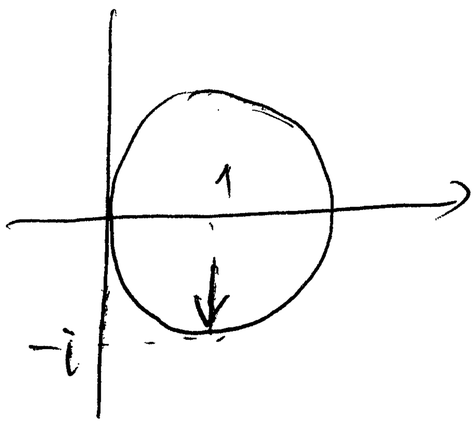
Then $f_1(z) \longrightarrow \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \dots\right) + i\frac{\pi}{4}$ as $z \to 1+i$ vertically up

Since $f$ is continuous, $f(1+i) = \lim_{z\to 1+i} f(z) = \lim_{\substack{z\to 1+i \\ \text{vertically up}}} f_1(z) =$

$$= \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots\right) + i\frac{\pi}{4}$$

Using the same technique for $1-i$ instead of $1+i$, we have

$$f(1-i) = \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots\right) - i\frac{\pi}{4}$$




(ii) We notice that for each $z \in \Omega$, $f(z) = f^+(z_1) + f^+(z_2)$ for some $z_1, z_2 \in \Omega^+$. ~~Thus~~ such that $z = z_1 z_2$

$$\begin{aligned}
\operatorname{Re} f(z) &= \operatorname{Re} f^+(z_1) + \operatorname{Re} f^+(z_2) \\
&= \cancel{\operatorname{Re} f^+} \log|z_1| + \log|z_2| \qquad \text{(by (vii) of Step 2)} \\
&= \log|z_1||z_2| \\
&= \log|z|
\end{aligned}$$

We have $\operatorname{Re} f(1+i) = \log|1+i| = \log\sqrt{2}$ Thus

$$f(1+i) = \log\sqrt{2} + i\frac{\pi}{4}$$

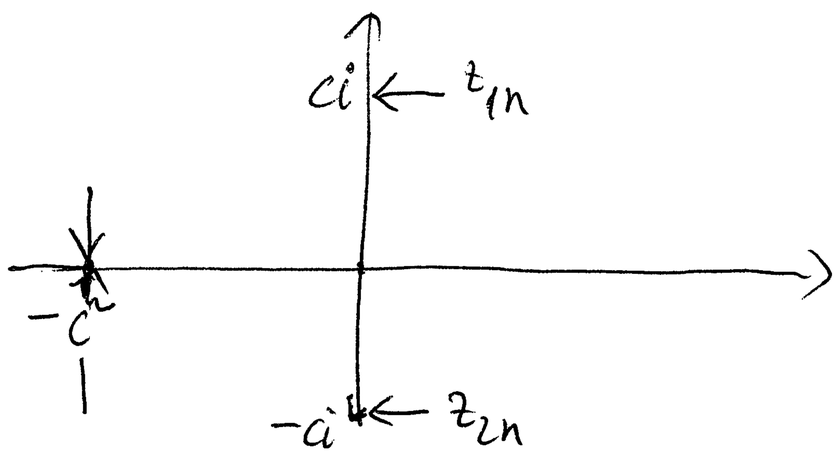
Similarly $f(1-i) = \log\sqrt{2} - i\frac{\pi}{4}$ We have

$$f\left(\frac{1+i}{\sqrt{2}}\right) = f(1+i) - f(\sqrt{2}) = \left(\log\sqrt{2} + i\frac{\pi}{4}\right) - \log\sqrt{2} = i\frac{\pi}{4}$$

and then $f(i) = f\left(\left(\frac{1+i}{\sqrt{2}}\right)^2\right) = 2f\left(\frac{1+i}{\sqrt{2}}\right) = 2i\frac{\pi}{4} = i\frac{\pi}{2}$

Similarly, $f(-i) = -i\frac{\pi}{2}$.

(iii) Let $c > 0$, we show that the limits of $f(z)$ as $z$ approaches $-c^2$ downward (path 1) and upward (path 2) are different.




Put $z_{1n} = \frac{1}{n} + ci$ and $z_{2n} = \frac{1}{n} - ci$.

We have $z_{1n} \to ci$, $z_{2n} \to -ci$, and

$$f(z_{1n}) \to f(ci) = f(c) + f(i) = f(c) + i\frac{\pi}{2},$$

$$f(z_{2n}) \to f(-ci) = f(c) + f(-i) = f(c) - i\frac{\pi}{2}$$

We have $z_{1n}^2 \to -c^2$, $z_{2n}^2 \to -c^2$ and

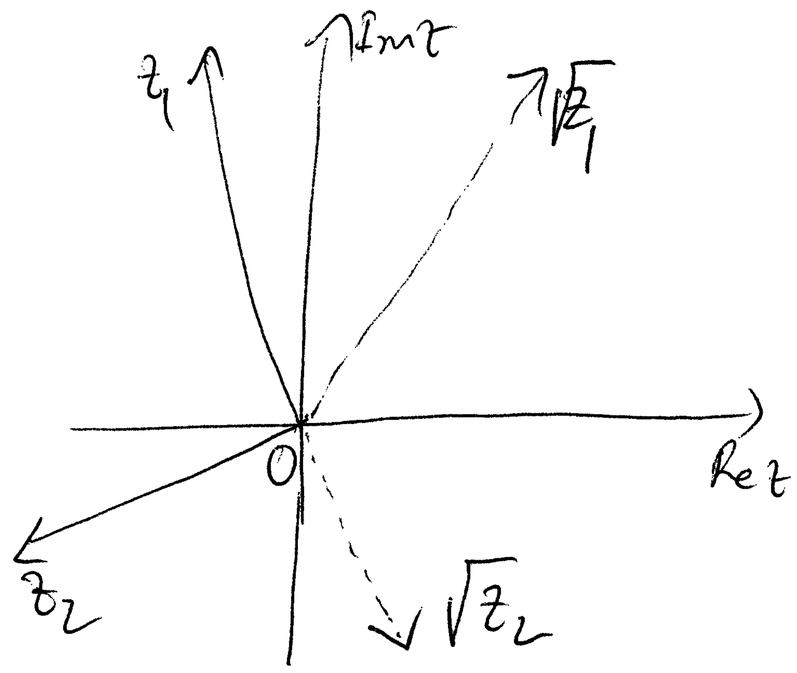
$$f(z_{1n}^2) = 2f(z_{1n}) \to 2f(ci) = 2f(c) + i\pi$$

$$f(z_{2n}^2) = 2f(z_{2n}) \to 2f(-ci) = 2f(c) - i\pi$$

Thus $f$ cannot be extended continuously at $z = -c^2$

Step 5 We'll show that $f$ is injective from $\Omega \to \mathbb{C}$.

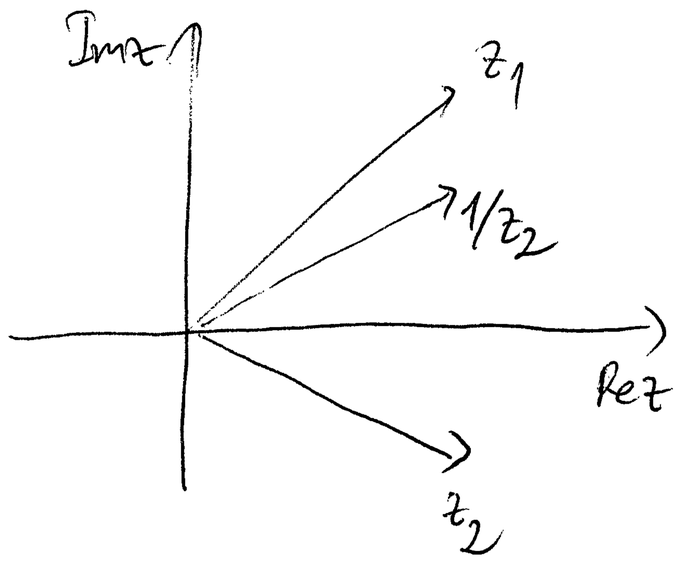
Suppose that there exist $z_1 \neq z_2$ in $\Omega$ such that $f(z_1) = f(z_2)$.




There exists a square root of $z_1$ that belongs to $\Omega^+$, which is denoted by $\sqrt{z_1}$.

Also, there exists a square root of $z_2$ that belongs to $\Omega^+$, which is denoted by $\sqrt{z_2}$.

We have $f(\sqrt{z_1}) = \frac{1}{2} f(z_1) = \frac{1}{2} f(z_2) = f(\sqrt{z_2})$ and $\sqrt{z_1} \neq \sqrt{z_2}$. Thus, ~~we can assume that~~ by replacing $z_1$ by $\sqrt{z_1}$, $z_2$ by $\sqrt{z_2}$, we can assume that $z_1, z_2 \in \Omega^+$. Then $1/z_2 \in \Omega^+$, and $z_0 = z_1/z_2 \in \Omega$. We have by the

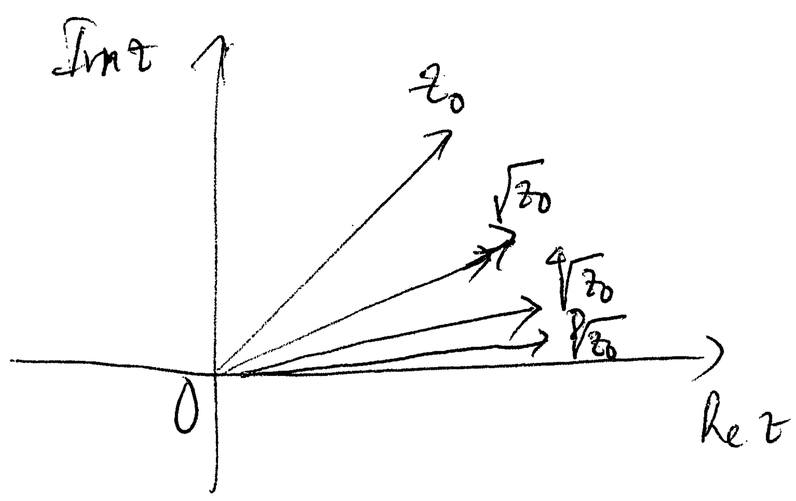



definition of $f(z)$:

$$f(z_0) = f^+(z_1) + f^+\left(\frac{1}{z_0}\right) = f^+(z_1) - f^+(z_2) = 0$$

Again, by replacing $z_0$ by a square root of $z_0$ that lies in $\Omega^+$, we can assum that $z_0 \in \Omega^+$.

Now we have $z_0 \in \Omega^+$, $z_0 \neq 1$ and $f(z_0) = 0$.




We have $0 = f(z_0) = \operatorname{Re} f(z_0) = \log|z_0|$.

Thus $|z_0| = 1$. We can choose one square root of $z_0$ which lies in the same quadrant as that of $z_0$, which is denoted $\sqrt{z_0}$. Again, we can choose one square root of $\sqrt{z_0}$ which lies in the same quadrant as that of $\sqrt{z_0}$, which is denoted by $\sqrt[4]{z_0}$, ....

We denote $z_0 = z_0$, $z_1 = \sqrt{z_0}$, $z_2 = \sqrt[4]{z_0}$, $z_3 = \sqrt[8]{z_0}$, ... Then the sequence $(z_n)$ has the property $z_m \neq z_n$ $\forall m \neq n$. ~~Indeed, suppose that there are $m < n$ such that $z_m = z_n$. Then $(z_m)^{2^m}$~~ and $z_n \to 1$ as $n \to \infty$ (we are using intuitive proof, not rigorous!). Moreover,

$$0 = f(z_0) = f(z_m^{2^m}) = 2^m f(z_m), \quad \text{and hence } f(z_m) = 0$$

Therefore
$$f'(1) = \lim_{m\to\infty} \frac{f(z_m) - f(1)}{z_m - 1} = 0$$

This is a contradiction since $f'(z) = \frac{1}{z} \neq 0, \forall z \in \Omega$

Step 6 We show that the range of $f$ is $\{w : -\pi < \operatorname{Im}(w) < \pi\}$

We define $g : \Omega \longrightarrow \mathbb{R}$
$$g(z) = \operatorname{Im} f(z)$$

Since $f$ is continuous, $g$ is also continuous. Since $\Omega$ is connected, the range of $g$ must be <u>connected</u>. Thus it is an interval in $\mathbb{R}$. Look at the sequence $\{f(z_{1n}^2)\}$ and $\{f(z_{2n}^2)\}$. We see that

$$f(z_{1n}^2) \longrightarrow 2f(c) + i\pi = 2\log c + i\pi \quad , \text{ thus } \operatorname{Im} f(z_{1n}^2) \to \pi$$

$$f(z_{2n}^2) \longrightarrow 2f(c) - i\pi = 2\log c - i\pi \quad , \text{ thus } \operatorname{Im} f(z_{2n}^2) \to -\pi$$

Therefore $(-\pi, \pi)$ is contained in the range of $g$. Suppose that the range of $g$ is not $(-\pi, \pi)$, then it must contain $-\pi$ or $\pi$. WLOG, we can assume $\pi$ is in the range of $g$. Then there exist $z_0 \in \Omega$ such that $\operatorname{Im} f(z_0) = \pi$.

We have $\operatorname{Im} f(z_0/|z_0|) = \operatorname{Im}(f(z_0) - \log|z_0|) = \operatorname{Im} f(z_0) = \pi$.

Thus we can assume $|z_0| = 1$. Let $\sqrt{z_0}$ be one square root of $z_0$. Then

$$\cancel{f(\sqrt{z_0}) = \frac{1}{2} f(z_0) = \frac{1}{2} i\pi \Longrightarrow \mathcal{I}}$$

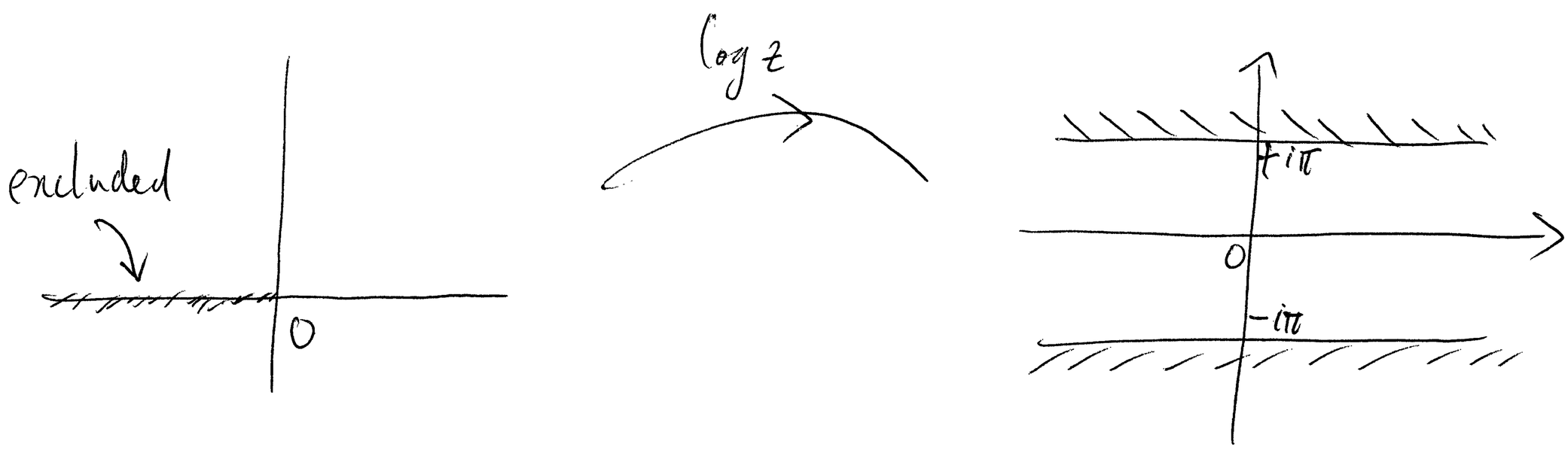
$$f(\sqrt{z_0}) = \log|\sqrt{z_0}| + i \operatorname{Im} f(\sqrt{z_0}) = \log 1 + i \operatorname{Im} \tfrac{1}{2} f(z_0) = i \tfrac{1}{2} \operatorname{Im} f(z_0)$$

$$= \frac{\pi}{2} i = f(i)$$

Then $\sqrt{z_0} = i$ because $f$ is injective (step 5). Then $z_0 = (\sqrt{z_0})^2 = i^2 = -1$.

This is a contradiction because $z_0 \in \Omega$.

Step 7 Now we can conclude that if the target set of $f$ is restricted to $\{w : -\pi < \operatorname{Im} w < \pi\}$ then $f$ is a bijection, which is from now denoted by $\log z$.




Let $g : \{z : -\pi < \operatorname{Im} z < \pi\} \longrightarrow \mathbb{C} \setminus \{w : \operatorname{Im} w = 0, \operatorname{Re} w \leq 0\}$ be the inverse of $\log z$ Then $g$ is analytic ~~any~~ and by chain rule:

$$1 = (z)' = (\log g(z))' = g'(z) \frac{1}{g(z)}$$

Then $g'(z) = g(z)$. Moreover, since $f(1) = 0$, we have $g(0) = 1$.

The function $g$ will be then denoted by $e^z$, which satisfies the ODE

$$\begin{cases} g'(z) = g(z) \\ g(0) = 1 \end{cases} \qquad \forall z \in \{w : -\pi < \operatorname{Im} w < \pi\}$$