

$$= \frac{1}{2h^\alpha} \left[|(T_h - I)u(x)| + \left| \frac{(T_h - I)^2}{h^2} u(x) \right| h^\alpha \right]$$

$$\Rightarrow \frac{|u(x+h) - u(x)|}{h^\alpha} \leq \frac{1}{2h^\alpha} ([u]_{\alpha, \alpha} 2h^\alpha + N)$$

$$\sup | [u]_{\alpha, \alpha} \leq \underbrace{2^{\alpha-1}}_{< 1 \text{ only if } 0 < \alpha < 1} [u]_{\alpha, \alpha} + N M_{\alpha, \alpha}^{(2)}$$

$$\Rightarrow [u]_{\alpha, \alpha} \leq N M_{\alpha, \alpha}^{(2)}$$

Application

$$f \in C_0^\alpha, g \in C_0^\beta, 0 < \alpha, \beta < 1, \alpha + \beta \neq 1$$

$$(\text{Note: } C^{1+\varepsilon} := C^{1, \varepsilon})$$

$$\Rightarrow f * g \in C^{\alpha+\beta}$$

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y) g(x-y) dy$$

$$\delta_*(f * g) = (\delta_* f) * (\delta_* g)$$

01/22/2014

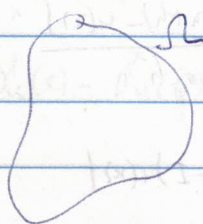
Classical Mechanics: describe motion

$$\circ \ddot{x}^{(1)}(t)$$

$$\ddot{x}^{(i)}(t) = F(x^{(1)}, \dots, x^{(n)})$$

$$\circ x^{(n)}(t)$$

Fluids: viewed as a continuum



Two ways to describe the motion of fluids.

1. Eulerian description: (1757)

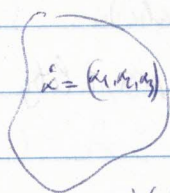
velocity field $u(x, t)$.

Individual particle trajectory: $\frac{dx}{dt} = u(x(t), t)$.

$u(x,t)$ is the main variable to be studied.

2. Lagrangian

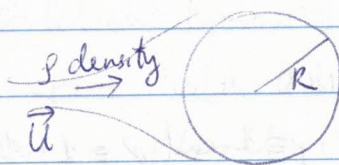
(x_1, x_2, x_3) : coordinates of a particle at time $t=0$.



These can be viewed as the labels for particles.

Position function: $\phi(x,t) = \phi^t(x)$

Velocity: $u(\phi^t(x), t) = \frac{d}{dt} \phi^t(x)$



According to Newton:

drag force $F = c \frac{1}{2} \rho R^2 U^2$ true for very large U .

According to Stokes: $F = 6\pi \mu R U$ true for small U .
(1850s) $\uparrow$ viscosity

How about the drag force for general velocity? Do we need to understand the motion of every particle? Somehow yes.



pingpong ball

why is the pingpong ball stable if we blow air/water, but not if we blow sand?

Compare with the spinning top:



3 degrees of freedom. It's hard to explain the phenomenon of why the ball is not dropping as it is spinning from Newton's second law $F=ma$.

Fluids have infinite degrees of freedom!

Equations of motion

Incompressible fluids of constant density ρ .

$u(x,t)$: velocity in $\mathbb{R}^3$

$p(x,t)$: pressure

$f(x,t)$: exterior force, ν viscosity (kinematic)

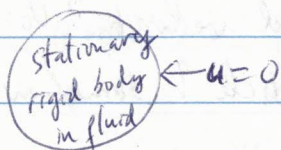
1) $\operatorname{div}(u) = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0$: incompressibility condition

2) $\frac{\partial u}{\partial t} + \underbrace{(u \cdot \nabla)u} + \frac{1}{\rho} \nabla p - \nu \Delta u = f(x,t)$

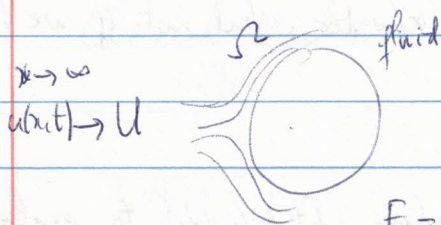
$$\nabla u = \left(\sum_{j=1}^3 u_j \frac{\partial u_i}{\partial x_j} \right)_{i=1,2,3}$$

We can choose units so that $\rho \equiv 1$ and $\nu \equiv 1$. Then the equation becomes $u_t + (u \cdot \nabla)u + \nabla p - \Delta u = f(x,t)$.

Boundary condition $u = 0$ on $\partial\Omega$



Return to the problem of drag force:



$$\begin{cases} u_t + (u \cdot \nabla)u + \nabla p - \Delta u = f \\ \operatorname{div}(u) = 0 \\ u(x,t) = 0 \text{ on } \partial\Omega \end{cases}$$

This is a very complicated problem, even for numerical calculation

01/24/2014

$$\begin{cases} u_t + (u \cdot \nabla)u + \nabla p - \Delta u = f \\ \operatorname{div} u = 0 \end{cases} \quad \text{in } \Omega \times (t_0, t_2)$$

$(\Omega = \mathbb{R}^n)$

Symmetries

• Obvious ones:

$$(i) \quad u(x, t) \rightarrow u(x-a, t)$$

$$p(x, t) \rightarrow p(x-a, t)$$

$$(ii) \quad R \in SO(n).$$

$$u(x, t) \rightarrow Ru(R^{-1}x, t)$$

$$p(x, t) \rightarrow p(R^{-1}x, t)$$

(iii) Galilean symmetry

$$\tilde{u}(x, t) = u(x + Ut, t) - U,$$

$$\tilde{p}(x, t) = p(x + Ut, t)$$

$$\frac{\partial \tilde{u}}{\partial t} = \text{old term} + U \cdot \nabla u,$$

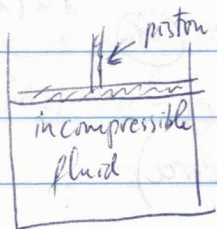
+

$$(\tilde{u} \cdot \nabla) \tilde{u} = \text{old term} - U \cdot \nabla u$$

doesn't change

$$(iv) \quad u, p \rightarrow u, p + \gamma(t), \text{ where } \gamma(t) \text{ is any func of time}$$

A serious weakness of Navier-Stokes equations:

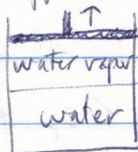


If we push the piston down, nothing will happen because the fluid is incompressible.

$$u \equiv 0$$

$$p = \text{const} = \gamma(t).$$

If we lift the piston up, the ~~no~~ equations predict that nothing would happen either. In reality however, there appear bubbles in the ~~fluid~~ fluid.



This happens especially when the pressure is low.

(v) Scaling symmetry (Reynolds 1880s)

$$u(x,t) \rightarrow \lambda u(\lambda x, \lambda^2 t),$$

$$p(x,t) \rightarrow \lambda^2 p(\lambda x, \lambda^2 t).$$

$$f(x,t) \rightarrow \lambda^3 f(\lambda x, \lambda^2 t)$$

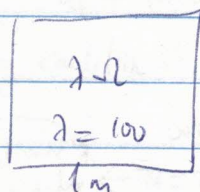
Compare with the heat equation: $u_t - \Delta u = 0$. We have the symmetry

$$u(x,t) \rightarrow \alpha u(x,t) + \beta \quad (\text{Celsius to Fahrenheit})$$

$$u(x,t) \rightarrow u(\lambda x, \lambda^2 t)$$

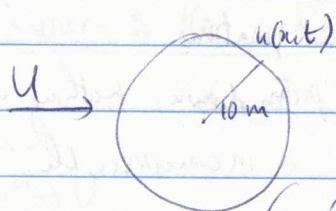
Application

 $u(x,t)$ cools in minutes



$u\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right) \Rightarrow$ the time to cool is lowered down by $\lambda^2 = 10000$ times.

Similar to Navier-Stokes



$10U \rightarrow \lambda u(\lambda x, \lambda^2 t)$

(to have the same drag force)

Cauchy problem in $\mathbb{R}^n$

$t \uparrow$

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0$$

$u(x,0) = u_0 \rightarrow$

$x = (x_1, \dots, x_n)$

$u_0(x)$ - initial condition

There was a following claim: $\|u\|_{L^q} < \infty$ for some certain q

then the Cauchy problem has a solution".

$$\begin{array}{ccc} u(x,t) & \longrightarrow & \lambda u(\lambda x, \lambda^2 t) = u_\lambda(x,t) \\ u_0(x) & & \lambda u_0(\lambda x) = u_{0,\lambda}(x) \end{array}$$

then for $n=3$,

$$\|u_{0,\lambda}\|_{L^2}^2 = \frac{1}{\lambda} \|u_0\|_{L^2}^2$$

$$\Rightarrow \|u_{0,\lambda}\|_{L^2} = \frac{1}{\sqrt{\lambda}} \|u_0\|_{L^2}$$

$\Rightarrow$ we can always find $\lambda > 0$ such that $\|u_{0,\lambda}\|_{L^2} < \varepsilon$.

If the claim is true then $\exists$ global solution u_λ . Then unscaling u_λ to get a global solution $u \Rightarrow$ the claim is suspicious!

In general

$$\|u_{0,\lambda}\|_{L^p} = \lambda^{1-\frac{n}{p}} \|u_0\|_{L^p}$$

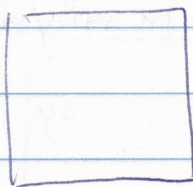
If $p=n$ then $\|u_{0,\lambda}\|_{L^p} = \|u_0\|_{L^p} \quad \forall \lambda > 0$.

Claim 2

If $\|u_0\|_{L^n} < \varepsilon_1$ then $\exists$ global solution.

This is a more reasonable claim, and in fact is true! We'll prove it in this course.

Meteorology model



$$u(x,t) \longrightarrow \lambda u(\lambda x, \lambda^2 t)$$

time scale = days

distance scale = 1000 km

$\Rightarrow$ the meteorological change can be viewed in a swimming pool, but then the velocity and get extremely huge.

01/29/2014 Compare Reynolds's symmetry with Dimensional Analysis.
 see the first lecture in the lecture notes of Topics in Math Physics,
 2011-2012.

Linear Stokes
$$\begin{cases} u_t - \Delta u + \nabla p = f \\ \operatorname{div} u = 0 \end{cases} \quad (\text{I})$$

Heat equation
$$\begin{cases} u_t - \Delta u = f \\ u(x, 0) = u_0 \end{cases} \quad \mathbb{R}^n \times [0, \infty) \quad (\text{II})$$

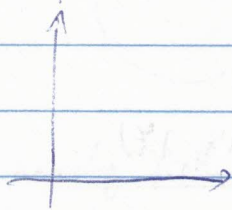
"Heat kernel"
$$P(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$$

Solution to (II) is
$$u(x, t) = \int_{\mathbb{R}^n} P(y, t) u_0(y) dy + \int_0^t \int_{\mathbb{R}^n} P(y, s) f(y, t-s) dy ds \quad (1)$$

Short hand: (if we write $u(t) = u(\cdot, t)$, $f(t) = f(\cdot, t)$, ...)

$$u(t) = P(t) * u_0 + \int_0^t P(t-s) * f(s) ds$$

Definition of what is a solution of the Cauchy problem.



Tempting definition: (f, u_0 smooth)

A smooth function u satisfies

$$\begin{cases} u_t - \Delta u = f \\ u(x, 0) = u_0(x) \end{cases}$$

A Problem is that the problem may not have a unique solutions.

Counterexample by Tykhonov (1930s)

$$\begin{cases} \partial_t u - \Delta u = 0 \\ u(x, 0) = 0 \end{cases} \quad \text{but } u \not\equiv 0.$$

We need that u be in a class which gives uniqueness.

One possible solution: u_0 is bounded, f is bounded in x .

$$\sup_x |f(x,t)| \leq \theta(t)$$

continuous

→ add the requirement that $\sup_x |u(x,t)| \leq \theta_1(t)$
continuous

→ give uniqueness.

Theorem If u_0 and f are smooth, bounded, then $\exists!$ solution u in the class " $\sup_x |u(x,t)| \leq \theta_1(t)$ - continuous". The solution is given by (1).

In case f is not smooth, $u_0 \equiv 0$:

$$u(t) = \int_0^t \Gamma(t-s) * f(s) ds$$

how does this satisfy the equation? because $f \in C_{\text{cont}} \nrightarrow u \in C_t^1 C_x^2$.

Def u satisfies $\underbrace{u_t - \Delta u = f}_{(H)}$ weakly in $\mathcal{O} \subset \mathbb{R}^n \times \mathbb{R}$ if

$$\iint u(-\varphi_t - \Delta \varphi) = \iint f \varphi \quad \forall \varphi \in \mathcal{D}(\mathcal{O})$$

Weak solution of the Cauchy problem

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^n} u(-\varphi_t - \Delta \varphi) dx dt &= \\ &= \int_0^\infty \int_{\mathbb{R}^n} f \varphi dx dt + \int_{\mathbb{R}^n} u_0 \varphi dx \end{aligned}$$

Uniqueness of weak solutions

1) Definition makes sense for $u \in L_{\text{loc}}^1(\mathbb{R}^n \times [0, \infty))$

→ need additional assumption.

2) Example of a uniqueness class

$\forall t_1 > 0$, $u \in L^2(\mathbb{R}^n \times (0, t_1))$ and is a weak solution.

⇒ uniqueness, given by the representation form

$$(u_0 \in L^\infty, f \in L^\infty(\mathbb{R}^n \times (0, t_1)) \forall t_1 > 0)$$

Regularity of the weak solutions:

Examples:

(1) "Weyl's lemma" for heat equations:

$\Omega \subset \mathbb{R}^n \times \mathbb{R}$ open, $u \in L^1_{loc}(\Omega)$ and $\iint u(-\partial_t - \Delta \varphi) dx dt = 0 \quad \forall \varphi \in C_c^\infty(\Omega)$

then $u \in C^\infty$.

01/31/2014

Linear Stokes equation

$$\begin{cases} u_t + \nabla p - \Delta u = f \\ \operatorname{div} u = 0 \\ u(x, 0) = u_0(x) \end{cases} \quad \text{in } \mathbb{R}^n \times (0, \infty)$$

① Representation formula for solutions:

Case 1: $f(x, t) = \nabla \phi(x, t)$, $u_0 \equiv 0$

Then a solution is $u \equiv 0$, $p = \phi$. Other solutions of this form

$$p = \phi + \theta(t)$$

Case 2: $f \equiv 0$, $\operatorname{div} u_0 = 0$

$u(t) = \Gamma(t) * u_0$, $p \equiv 0$, where Γ is the heat condition kernel.
↑
spatial convolution

Note that $\operatorname{div}(\Gamma(t) * u_0) = \Gamma(t) * (\operatorname{div} u_0) = 0$.

Remark: This works only if there is no boundary, for example

$$\Omega = \mathbb{R}^n, \quad \Omega = \mathbb{R}^n / \mathbb{Z}^n$$

Case 3 $u_0 \equiv 0$, $\text{div} f = 0$

$$u(t) = \int_0^t \Gamma(t-s) * f(s) ds, \quad p \equiv 0$$

Note that for a domain with boundary, if $\text{div} u_0 = 0$ then $\text{div} u$ is not necessarily zero for $t > 0$.

General case = linear combination of cases 1, 2, 3 because of the * Helmholtz decomposition:

$$\begin{array}{c} f(x,t) \\ \uparrow \\ \text{general} \end{array} = g(x,t) + \nabla \phi(x,t)$$

$\text{div} g(x,t) = 0$

A way to view the Helmholtz decomposition:

$D(\mathbb{R}^n, \mathbb{R}^n)$ - smooth, compactly supported, vector fields in $\mathbb{R}^n$.

$$X = \{v \in D(\mathbb{R}^n, \mathbb{R}^n), \text{div} v = 0\}$$

$$Y = \{\nabla \phi, \phi \in \mathcal{L}(\mathbb{R}^n) \text{ - smooth, compactly supported, scalar function}\}$$

Observation: $X \perp Y$ in $L^2(\mathbb{R}^n, \mathbb{R}^n)$.

$$(u, v) = \int_{\mathbb{R}^n} \sum_{i=1}^n u_i(x) v_i(x) dx$$

Proof of $X \perp Y$:

$$\int_{\mathbb{R}^n} u \nabla \phi dx = \int_{\mathbb{R}^n} u_i \frac{\partial \phi}{\partial x_i} dx = - \int_{\mathbb{R}^n} \frac{\partial u_i}{\partial x_i} \phi dx = 0$$

Claim: $L^2(\mathbb{R}^n; \mathbb{R}^n) = \overline{X} \oplus \overline{Y}$ (closure in L^2).

Proof suffices to show that if $f \in L^2$, $f \perp X$, $f \perp Y$ then $f = 0$.

It suffices to assume that f is smooth (by replacing f with $f * \eta_\epsilon$)

$$f \perp \nabla \phi \quad \forall \phi$$

$$\Rightarrow \int_{\mathbb{R}^n} f \nabla \phi dx = \int_{\mathbb{R}^n} (\text{div} f) \phi dx = 0 \quad \forall \phi \Rightarrow \text{div} f = 0.$$

we know that $\operatorname{div}(\operatorname{curl} \psi) = 0$. Thus,

$$\int_{\mathbb{R}^n} f \operatorname{curl} \psi - \int_{\mathbb{R}^n} (\operatorname{curl} f) \psi = 0 \quad \forall \psi$$

$$\Rightarrow \operatorname{curl} f = 0$$

$$\begin{cases} \operatorname{div} f = 0 \\ \operatorname{curl} f = 0 \end{cases} \Rightarrow \Delta f = 0$$

Since $f \in L^2(\mathbb{R}^n)$, $f \equiv 0$ (Liouville's theorem).

$P: L^2 \rightarrow X$ orthogonal projection.

$$f = \underbrace{Pf}_g + \underbrace{(I-P)f}_{\nabla \phi}$$

Return to the general case:

$$\begin{cases} u_t - \Delta u + \nabla p = f = Pf + (I-P)f \\ \operatorname{div} u = 0 \\ u(x, 0) = u_0(x) \end{cases}$$

Abstract representation formula

$$u(t) = \Gamma(t) * u_0 + \int_0^t \Gamma(t-s) * (Pf)(s) ds$$

More explicit representation

Expect: $u(t) = K(t) * u_0 + \int_0^t K(t-s) * f(s) ds$

$$u_i(t) = \sum_j K_{ij}(t) * u_{0j} + \int_0^t \sum_j K_{ij}(t-s) * f_j(s) ds$$

Example: $u_t - \Delta u + \nabla p = b(t)$ - independent of x .

Solution 1: $p = 0$, $u = u(t)$ (x -independent)

$$\frac{du}{dt}(t) = b(t)$$

Solution 2 $u \equiv 0$, $p(x,t) = b(t) \cdot x$

Note: the Helmholtz decomposition works for function vector fields in $L^1(\mathbb{R}^n) \cap L^\infty \cap L^p(\mathbb{R}^n)$, $1 < p < \infty$.

$$u_t - \Delta u + \nabla p = 0$$

$$h = h(x,t), \quad \Delta_x h = 0, \quad \text{any dependence on } t.$$

$$u = \nabla_x h, \quad p = -h_t$$

This is an example to show that Weyl's lemma is not true (local solution is not smooth).

02/03/2014

$$f = \underbrace{g}_{\text{div } g = 0} + \nabla \phi \quad \Rightarrow \quad \text{div } f = \Delta \phi$$

G = fundamental sol. of Laplace problem, $G(x) = -C \frac{1}{|x|^{n-2}}$

$$\phi = G * \text{div } f = G * f_{,ij} = G_{,ij} * f_j$$

$$G_{,ij} = C_n \frac{x_j}{|x|^n} \Rightarrow G_{,ij} \in L^1_{loc}$$

$$\phi_{,i} = (G_{,j} * f_j)_{,i} = G_{,ij} * f_j$$

Away from 0: $G_{,ij} = C_n \left(\frac{\delta_{ij}}{|x|^n} - n \frac{x_i x_j}{|x|^{n+2}} \right)$
(-n)-homogeneous

$$G_{,ij}^\varepsilon(x) = \begin{cases} G_{,ij} & \varepsilon < |x| < \varepsilon^{-1} \\ 0 & \text{elsewhere} \end{cases}$$

$$\underbrace{I_{ij} \phi}_{\text{SIO}} = \lim_{\varepsilon \rightarrow 0} G_{,ij}^\varepsilon * \phi$$

SIO (singular integral operator)

$$\int_{S^{n-1}} G_{ij}(x) dx = 0$$

$$G_{ij} * \phi = I_{ij} \phi + \delta * \phi$$

therefore $(Pf)_j = f_j - \underline{G_{ij}} * f_j$

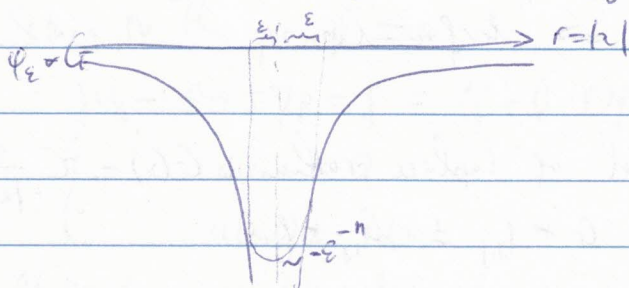
distribution, not singular integral operator

$$= f_j - I_{ij} \phi - \delta * \phi$$

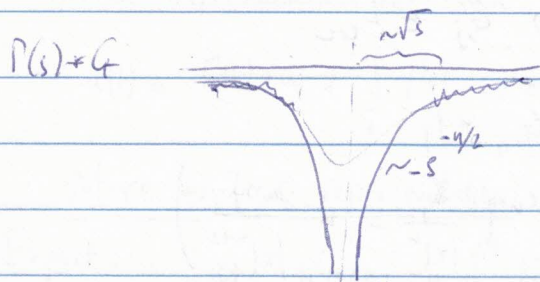
We have

$$\begin{aligned} \Gamma(s) * (Pf)_j(t-s) &= \Gamma(s) * (f_j - \underline{G_{ij}} * f_j)(t-s) \\ &= \Gamma(s) * f_j(t-s) + \Gamma(s) * (\underline{G_{ij}} * f_j)(t-s) \\ &= \Gamma(s) * f_j(t-s) + (\Gamma(s) * \underline{G_{ij}}(s)) * f_j(t-s) \end{aligned}$$

We have $\Gamma(s) * \underline{G_{ij}}(s) = (\Gamma * \underline{G})_{ij}$. Put $\Phi(x,s) = \Gamma(s) * \underline{G}$.



$$\psi_\varepsilon(x) = \varepsilon^{-n} \psi\left(\frac{x}{\varepsilon}\right), n=3$$



Note that $\Gamma(s)$ doesn't have compact support.

$$\Gamma(s) * G_{ij} = (\Phi(\cdot, s))_{,i x_j}$$

Scaling of $\Phi(x,s)$:

$$\begin{cases} \Phi_t - \Delta \Phi = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ \Phi(x, 0) = G \end{cases}$$

$$\lambda^{-(n-2)} G(\lambda x) = G(x) \Rightarrow \lambda^{-(n-2)} \Phi(\lambda x, \lambda^2 s) = \Phi(x, s)$$

$$\lambda = \frac{1}{\sqrt{s}} : \Phi(x, s) = \frac{1}{s^{\frac{n-2}{2}}} \Phi\left(\frac{x}{\sqrt{s}}, 1\right)$$

$$\Rightarrow \Phi(x, s) = \frac{1}{s^{\frac{n-2}{2}}} M\left(\frac{x}{\sqrt{s}}\right) \quad (n > 2)$$

where $M(r) \sim \frac{C_n}{(1+|x|^2)^{\frac{n-2}{2}}}$

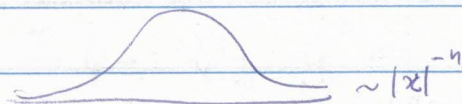
We have
$$u_i(t) = \int_0^t \Gamma(s) * f_i(t-s) ds + (\Gamma(s) * G)_{,ij} * f_j(t-s) ds$$

$$= \int_0^t K_{ij}(s) * f_j(t-s) ds$$

$$K_{ij}(x, t) = \delta_{ij} \Delta \Phi(x, t) - \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x, t)$$

$$K_{ij}(x, t) = \frac{1}{t^{\frac{n}{2}}} M_{ij}\left(\frac{x}{\sqrt{t}}\right)$$

M_{ij} has slow decay at infinity



Convergent

Heat kernel: $\Gamma(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} = \frac{1}{t^{n/2}} F\left(\frac{x}{\sqrt{t}}\right)$, where

$$F(x) = \frac{1}{(4\pi)^{n/2}} e^{-\frac{|x|^2}{4}}$$

$M_{ij} \sim |x|^{-n}$ at infinity, K_{ij} is not integrable in $\mathbb{R}^n$.

Example

$f(x, t)$ bounded

$$u(x, t) = \int_0^t \int_{\mathbb{R}^n} \Gamma(x, t) f(x-y, t-s) dy ds \quad \text{well-defined}$$

$$\int_0^t \int_{\mathbb{R}^n} K_{ij}(y, s) f_j(x-y, t-s) dy ds \quad \text{is not well-defined}$$

* Question What is the regularity of the function $\Phi(x, s) = (f(s) * G)(x)$?

02/05/2014

$$\Phi(x, s) = (f(s) * G)(x) = \frac{1}{s^{\frac{n-2}{2}}} F\left(\frac{x}{\sqrt{s}}\right)$$

where F is a smooth function. Thus Φ is smooth for $s > 0$.

$$u_i(t) = \int_0^t K_{ij}(s) * f_j(t-s) ds$$

$$f = f(x, t) \mapsto u = u(x, t)$$

How do the properties of f reflect the properties of u ?

1) Hölder inequality:

$$\int fg \leq \left(\int |f|^p \right)^{\frac{1}{p}} \left(\int |g|^q \right)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1$$

2) Young's inequality:

If $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, then $\|f * g\|_r \leq \|f\|_p \|g\|_q$ where

$$\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$$

Special case: $q = 1$, $\|f * g\|_p \leq \|f\|_p \|g\|_1$

$$\|f * g\|_\infty \leq \|f\|_p \|g\|_{p'}$$

Then interpolate to get $\|f * g\|_r \leq \|f\|_p \|g\|_q$

3) Fractional Interpolation:

$$f \in L^p(\mathbb{R}^n), \quad I_k f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-k}} dy$$

We have $\|I_k f\|_q \leq C_p \|f\|_p$, where $p > 1$, $\frac{1}{q} = \frac{1}{p} - \frac{k}{n} > 0$

→ Theorem 4.18, p.225, Bennett-Sharpley "Interpolation of Operators".

Idea of proof: Marcinkiewicz Interpolation.

Exercise Exercise. $f_\lambda(x) = f(\lambda x)$
 $(I_\lambda f_\lambda)(x) = \lambda^{-\kappa} (I_\lambda f)(\lambda x)$

$$\|f_\lambda\|_p = \lambda^{-\frac{n}{p}} \|f\|_p$$

We must have $\|I_\lambda f_\lambda\|_q \leq C_p \|f_\lambda\|_p$ for all $\lambda > 0$. Thus

we must have $\frac{1}{q} = \frac{1}{p} - \frac{\kappa}{n}$.

If $\frac{1}{p} - \frac{\kappa}{n} < 0$ then $|I_\lambda f(x) - I_\lambda f(y)| \leq C_p \|f\|_p$
 where $-\frac{\kappa}{n} = \frac{1}{p} - \frac{\kappa}{n}$ (??)

4) Calderon-Zygmund estimate:

I : singular integral operator

Eg the projection $P: L^2 \rightarrow L^2$ is a singular integral operator (mod a pointwise operator).

$$I_\varepsilon f = \lim_{\varepsilon \rightarrow 0} G_\varepsilon * f$$

truncated kernel

$$\|I_\lambda f\|_{L^p} \leq C_p \|f\|_{L^p}, \quad p \in (1, \infty)$$

For $f = f(x, t)$, we denote

$$\|f\|_{L_t^p L_x^q}^p = \left\{ \int_{-\infty}^{\infty} \left(\int |f(x, t)|^q dx \right)^{\frac{p}{q}} dt \right\}^{\frac{1}{p}} = \left\{ \int_{-\infty}^{\infty} \|f(x, t)\|_q^p dt \right\}^{\frac{1}{p}}$$

If $f(t)$ is used to denote $f(\cdot, t)$ then we can write

$$\|f\|_{L_t^p L_x^q}^p = \left(\int_{-\infty}^{\infty} \|f(t)\|_q^p dt \right)^{\frac{1}{p}}$$

$$u(t) = \int_0^t K_{ij}(s) * f_j(t-s) ds$$

$$\|u\|_{L_t^q L_x^p} \leq C \|f\|_{L_t^m L_x^r}$$

↗ assumptions

How to remember the relation between m, r and p, q ? use scaling:

$$u_t - \Delta u + \nabla p = f, \quad u(x, 0) = 0$$

$$u(\lambda x, \lambda^2 t) = u_x$$

$$\lambda^2 f(\lambda x, \lambda^2 t) = f_x$$

Short hand: $u(t) = \int_0^t K(s) * f(t-s) ds$

$$\text{Thus, } \|u(t)\|_{L_x^p} \leq \int_0^t \|K(s) * f(t-s)\|_{L_x^p} ds$$

$$\stackrel{\text{Young}}{=} \int_0^t \|K(s)\|_{L_x^a} \|f(t-s)\|_{L_x^r} ds \quad (*)$$

where $\frac{1}{p} + 1 = \frac{1}{a} + \frac{1}{r}$

Recall: $K(s) \sim \frac{1}{s^{\frac{n}{2}}} F\left(\frac{x}{\sqrt{s}}\right)$, $F \sim \frac{1}{(1+|x|^2)^{\frac{n}{2}}}$

Thus $K(s) \in L^a(\mathbb{R}^n)$ for any $a > 1$. We have

$$\begin{aligned} \|K(s)\|_a &= \frac{1}{s^{\frac{n}{2}}} \|F\left(\frac{\cdot}{\sqrt{s}}\right)\|_a \\ &= s^{-\frac{n}{2}} s^{\frac{1}{2}(\frac{n}{a})} \|F(\cdot)\| \\ &\sim s^{-\frac{n}{2}(1-\frac{1}{a})} \sim s^{-\frac{n}{2}\frac{1}{a'}} \end{aligned}$$

where $\frac{1}{a'} + \frac{1}{a} = 1$

Then

$$(*) \leq \int_0^t s^{\frac{-n}{2a'}} \|f(t-s)\|_{L_x^r} ds$$

Then use the fractional interpolation inequality ($n \geq 0$).

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$L_t^p L_x^q$: the space of all functions $t \mapsto \|f(t, \cdot)\|_q$ which is in $L^p(\mathbb{R})$. Last time:

$$u(t) = \int_0^t K(s) * f(t-s) ds$$

$$\|u(t)\|_p \leq C \int_0^t s^{\frac{-n}{2a'}} \|f(t-s)\|_r ds, \text{ where } \frac{1}{p} + 1 = \frac{1}{a'} + \frac{1}{r}$$

Since $a' > 1$, $p > r$. (which is consistent with the Marcinkiewicz theorem).

$t \mapsto \|u(t)\|_p$: function of t

Now calculate the L_t^q -norm of this function.

Assume $0 < \frac{n}{2a'} < 1$. Take $k_\beta = |s|^{-1+\beta}$, $\beta \in (0, 1)$, $n=1$

Apply the fractional inequality $k_\beta * g$ with

$$g(s) = \begin{cases} 0, & s \leq 0 \\ \|f(s)\|_r, & s > 0 \end{cases}$$

Then we get $\|k_\beta * g\|_{L_t^q} \leq C \|g\|_{L_t^m}$ with $\frac{1}{q} = \frac{1}{m} - \beta > 0$
Therefore,

$$\|u\|_{L_t^q L_x^p} \leq C \|f\|_{L_t^m L_x^r} \text{ where}$$

$$\begin{cases} \frac{2}{q} + \frac{n}{p} = \frac{2}{m} + \frac{n}{r} - 2 > 0 \\ p > r, q > m \end{cases}$$

In particular, $q = p$; $m = r$,

$$\|u\|_{L^{p,2}} \leq C \|f\|_{L^{p,r}}$$

where $\frac{1}{p} = \frac{1}{r} - \frac{2}{n+2} > 0$

Similarity with elliptic equations

$$\Delta u = f \text{ in } \mathbb{R}^n, \quad u = G * f$$

$$\begin{array}{ccc} f & \xrightarrow{\text{C.Z.}} \nabla^2 u & \xrightarrow{\text{Imbed.}} u \\ L^p & & L^q \end{array} \quad \frac{1}{q} = \frac{1}{p} - \frac{2}{n} > 0$$

fractional inequality

$$\begin{array}{ccccccc} & u \in L^q & & u \in C^\alpha & & u \in C^{1,\alpha} & \\ \underbrace{\hspace{1cm}} & & \underbrace{\hspace{1cm}} & & \underbrace{\hspace{1cm}} & & \\ p=1 & & p=\frac{n}{2} & & p=n & & p \in L^p \end{array}$$

Para. equation $u(0) = 0$

$$u(0) = 0$$

$$u_t - \Delta u = f$$

$$u_t - \Delta u + \nabla p = f$$

The gain of regularity from f to u ? $p = q$, $m = r$.

$$\begin{array}{ccccccc} & u \in L^{p,r} & & u \in C_{\text{par}}^\alpha & & \nabla u \in C_{\text{par}}^\alpha & \\ \underbrace{\hspace{1cm}} & & \underbrace{\hspace{1cm}} & & \underbrace{\hspace{1cm}} & & \\ r=1 & & r=\frac{n+2}{2} & & r=n+2 & & r \end{array}$$

$$C_{\text{par}}^\alpha \text{ means: } |u(x,t) - u(x',t')| \leq C(|x-x'|^2 + |t-t'|)^{\frac{\alpha}{2}}$$

$\Rightarrow$ the variable t seems to contribute 2 dimensions (or 2 variable)

Another way to look at this:

$$u(x,t) \rightarrow \mathcal{I}u(\lambda x, \lambda^2 t)$$

the exponent 2 suggests that t is counted
as two variables

Divergence form on the right-hand side

$$\Delta u = \frac{\partial f_j}{\partial x_j}, \quad u_t - \Delta u = \frac{\partial f_j}{\partial x_j}(x, t).$$

Stokes: $\frac{\partial u_t}{\partial t} - \Delta u + \nabla_i p = \frac{\partial}{\partial x_j} f_{ij}(x, t)$

Elliptic: $\Delta u = \frac{\partial f_j}{\partial x_j}$

$$u = G * (\frac{\partial f_j}{\partial x_j}) = G_{ij} * f_j$$

Regularity:

$$\underbrace{u \in L^p, \text{ where } \frac{1}{q} = \frac{1}{p} - \frac{1}{n} > 0}_{p=1} \quad \underbrace{u \in C^\alpha}_{p=n} \quad \rightarrow \quad f \in L^p$$

Parabolic: $u_t - \Delta u = \frac{\partial f_j}{\partial x_j}, \quad \text{Stokes} \dots$
 $u(0) = 0$

$$u_t(t) = \int_0^t K_{ij}(s) * (\frac{\partial f_j}{\partial x_j})_k(t-s) ds = \int_0^t K_{ij,k}(s) * f_{jk}(t-s) ds$$

(Recall: $K(t) = \frac{1}{t^{n/2}} F\left(\frac{x}{\sqrt{t}}\right), \quad F \sim \frac{1}{(1+|x|^2)^{n/2}}$)

$$\Rightarrow \frac{\partial K}{\partial x_i} \sim \frac{1}{t^{\frac{n+1}{2}}} \frac{\partial F}{\partial x_j} \left(\frac{x}{\sqrt{t}}\right) \quad \text{where} \quad \frac{\partial F}{\partial x_i} \sim \frac{1}{(1+|x|^2)^{\frac{n+2}{2}}} \in L^1_n$$

$$\underbrace{u \in L^p, \quad \frac{1}{p} = \frac{1}{r} - \frac{1}{n+2}}_{rp=1} \quad \underbrace{u \in C^\alpha_{par}}_{rp=n+2} \quad \rightarrow \quad f \in L^r_{t,x}$$

Note Non-divergence form: $u_t - \Delta u = f$

$$\frac{2}{q} + \frac{n}{p} = \frac{2}{m} + \frac{n}{r} - 2 > 0$$

Divergence form: $u_t - \Delta u = \operatorname{div} f$

$$\frac{2}{q} + \frac{n}{p} = \frac{2}{m} + \frac{n}{r} - 1 > 0$$

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$$u_t - \Delta u + \nabla p = \operatorname{div} f$$

$$u(t) = \int_0^t K'(t-s) * f(t-s) ds$$

$$[u]_{C_{par}^\alpha} = \sup \frac{|u(x,t) - u(x',t')|}{(|x-x'|^2 + |t-t'|)^{\alpha/2}}$$

Want to have an estimate:

$$[u]_{C_{par}^\alpha} \leq C \|f\|_{L_t^m L_x^r}$$

Note that $[\cdot]_{C_{par}^\alpha}$ is just a semi-norm. Thus, if $f \in L_t^m L_x^r$ then u may not be well-defined without additional assumptions

Elliptic case $[u]_{C^\alpha} = \sup \frac{|u(x) - u(x')|}{|x-x'|^\alpha}$

$$u_\lambda(x) = u(\lambda x), \quad [u_\lambda]_{C^\alpha} = \lambda^\alpha [u]_{C^\alpha}, \quad p = \lambda f_\lambda$$

$$-\Delta u = \operatorname{div} f$$

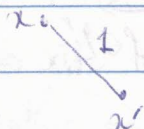
$$\Rightarrow -\Delta u_\lambda = \operatorname{div} f_\lambda$$

$$f \in L^p, p > n \Rightarrow \|\lambda f_\lambda\|_{L^p} = \lambda \cdot \lambda^{-\frac{n}{p}} \|f\|_{L^p} = \lambda^{1-\frac{n}{p}} \|f\|_{L^p}$$

If we want to have $[u]_{C^\alpha} \leq C \|f\|_p$, we must have $\alpha = 1 - \frac{n}{p}$.

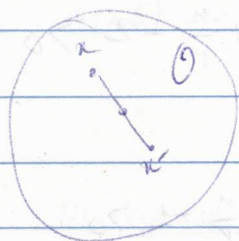
For each f and u , we can choose $\lambda > 0$ appropriate such that $|x-x'| = 1$.

(WLOG) want to find $C > 0$ such that



$$\int |K'(x-y)f(y) - K'(x'-y)f(y)| dy \leq C \|f\|_{L^p}$$

$$K' \sim \frac{1}{|x|^{n-1}}$$



$$\int_{\mathbb{R}^n} = \int_{\mathcal{O}} + \int_{\mathbb{R}^n \setminus \mathcal{O}}$$

For $\int_0$:

$$\int_0 |K(n-y)| \underbrace{|f(y)|}_{\in L^p} dy + \int_0 \underbrace{|K(n'-y)|}_{\text{the same}} \underbrace{|f(y)|}_{\text{the same}} dy$$

$s < \frac{n}{n-1}$

Note that $\frac{1}{p} = 1 - \frac{1}{p'} \ (p > n) > 1 - \frac{1}{n} = \frac{n-1}{n}$. Choose $s = p'$.

For $\int_{\mathbb{R}^n \setminus 0}$:

$$\int_0 \underbrace{|K(n-y) - K(n'-y)|}_{\sim O(|y|^{-n})} |f(y)| dy$$

$$\frac{1}{y^n} - \frac{1}{(y+1)^n} \sim \frac{1}{y^{n+1}}$$

$u(x,t)$

$f(x,t)$

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$$

$$\lambda f(\lambda x, \lambda^2 t)$$

want: $[u]_{C^{m,\alpha}_{pw}} \leq C \|f\|_{L^m_t L^r_x}$

$$\downarrow$$

$$\lambda f(\lambda x, \lambda^2 t)$$

$$\left\{ \begin{array}{l} u_\lambda \\ \lambda^\alpha \end{array} \right\}$$

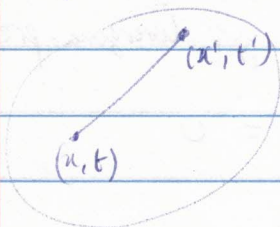
$$\downarrow$$

$$\lambda \cdot \lambda^{-\frac{2}{m} - \frac{n}{r}}$$

$\Rightarrow$ we must have $\alpha = 1 - \frac{2}{m} - \frac{n}{r} > 0$

Again, WLOG (with appropriate $\lambda > 0$), assume $|x-x'| = t$

$$(|x-x'|^2 + |t-t'|)^{\frac{1}{2}} = 1$$



$$|u(x,t) - u(x',t')| \leq C \|f\|_{L^m_t L^r_x} ?$$

$$\int_0 + \int_{\mathbb{R}^n \setminus 0}$$

Estimate $\int_0$ and $\int_{\mathbb{R}^n \setminus 0}$ the same way as in elliptic case.

Elliptic $f \in L^p, p > n$

$$u(x) = \int \frac{f(y)}{|x-y|^{n-1}} dy, \quad p > 0$$

It is easy to take f such that $u(x) = \infty \forall x$. Example, $f(y) = \frac{1}{(1+|y|)^{\frac{n}{p}}}$

$\Rightarrow u$ is not necessarily well-defined

$f_j \rightarrow f$ in L^p , f_j compactly supported

$$[u_j]_{C^\alpha} \leq C \|f_j\|_{L^p}$$

Usually, $\|u\|_{C^\alpha} = [u]_{C^\alpha} + \|u\|_{L^\infty}$

Question? How about the pressure p ?

$$u_t - \Delta u + \nabla p = f \in L_t^m L_x^q$$

Calderon-Zygmund: $\nabla p \in L_t^m L_x^q$

Sobolev imbedding $\Rightarrow p \in L_t^{\tilde{m}} L_x^{\tilde{q}}$

We can write a kernel for p .

02/12/2014

Divergence-free fields



$$\phi_\varepsilon(x) = \varepsilon^{-n} \phi(\varepsilon^{-1}x)$$

$$u_{\varepsilon,i}(x) = \int_0^L \gamma_i'(s) \phi_\varepsilon(\gamma(s) - x) ds$$

It's easy to check that u_ε is divergence-free.

$$u_{\varepsilon,i,c} = - \int_0^L \gamma_i'(s) \underbrace{\frac{\partial \phi_\varepsilon}{\partial x_i}(\gamma(s) - x)}_{\frac{d}{ds} \phi_\varepsilon(\gamma(s) - x)} ds = 0$$

At the boundary, $|u_\varepsilon| \sim \varepsilon^{-n+1}$
(8)

v : vector field



v is a gradient field $\Leftrightarrow \int_{\gamma} v_i dx_i = 0$
 γ closed curve γ .

$$\int \partial_i u_{\varepsilon_i} dx \xrightarrow{\varepsilon \rightarrow 0} \int_{\gamma} v_i dx_i$$

Claim: $u \in L^1_{loc}(\mathbb{R}^n, \mathbb{R}^n)$ is a distribution

$\forall \phi$ compactly supported, $\mathbb{R}^n \rightarrow \mathbb{R}^n$, smooth, $\text{div} \phi = 0$
 If $\int u_i \phi_i dx = 0 \quad \forall \phi$ then $u_i = \frac{\partial f}{\partial x_i}$

Ex: $n=2$, $u=(u_1, u_2)$, $\text{div} u = 0$

$$\text{curl} \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix} = 0$$

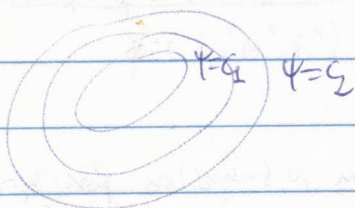
$$\Rightarrow u_1 = -\frac{\partial \psi}{\partial x_2}, \quad u_2 = \frac{\partial \psi}{\partial x_1}$$

ψ : stream function



flow through $\gamma = \psi(x)$

Divergence free field is more less a comp
 combination of closed level curves



Weak solution of $u - \Delta u + \nabla p = f$
 $\text{div} f$

Static: $-\Delta u + \nabla p = f$
 $\text{div} f$

1) Weak solution for Laplacian:

$$-\Delta u = \text{div} f$$

$$I(u) = \int \frac{1}{2} |u|^2 dx - \int f u dx \rightarrow \min$$

Characterized by $\frac{d}{d\varepsilon} I(u + \varepsilon \phi) = 0 \quad \forall \phi \in D(\mathbb{R}^n)$

Steady Stokes:

$$I(u) = \int \frac{1}{2} |Du|^2 + \nabla u \cdot f \xrightarrow{\text{min}} \quad \uparrow \quad \text{with constraint } \operatorname{div} u = 0$$

Weak solution: $\frac{d}{d\varepsilon} I(u + \varepsilon \phi) = 0 \quad \forall \phi, \operatorname{div} \phi = 0, \phi \in D(\mathbb{R}^n; \mathbb{R}^n)$

$$\int -\Delta u \cdot \phi - (\operatorname{div} f) \phi = 0$$

$$\Rightarrow \int (-\Delta u - \operatorname{div} f) \phi = 0 \quad \forall \phi \in D(\mathbb{R}^n; \mathbb{R}^n), \operatorname{div} \phi = 0$$

$$\Rightarrow -\Delta u - \operatorname{div} f = -\nabla p$$

$$\operatorname{div} u = 0$$

p serves as a Lagrange multiplier.

$$-\Delta u + \nabla p = \operatorname{div} f$$

L^2 -theory: $f \in L^2, Du \in L^2, p \in L^2$

Estimate: $\|Du\|_{L^2}^2 + \|p\|_{L^2}^2 \leq C \|f\|_{L^2}^2$

Stokes system with time



$$u_t - \Delta u + \nabla p = \operatorname{div} f$$

$$\operatorname{div} u = 0$$

$$u(x,0) = u_0(x)$$

Weak form: Same as for $u_t - \Delta u = \operatorname{div} f$, except take ϕ with $\operatorname{div} \phi = 0$.

$$\int_0^\infty \int_{\mathbb{R}^n} u(-\varphi_t - \Delta \varphi + f \cdot \nabla \varphi) dx dt = \int_{\mathbb{R}^n} u_0(x) \varphi(x, 0) dx \quad (*)$$

$$\forall \varphi \in D(\mathbb{R}^{n+1}), \operatorname{div}_x \varphi = 0.$$

Issues with uniqueness / smoothness

$$\left. \begin{array}{l} f \equiv 0 \\ u_0 \equiv 0 \end{array} \right\} \stackrel{?}{\Rightarrow} u \equiv 0$$

Fails for 2 reasons. 1) Tikhonov-type

$$\begin{cases} u_t - \Delta u = 0 \\ \operatorname{div} u = 0 \\ f \equiv 0, u_0(x) \equiv 0 \end{cases} \quad \begin{array}{l} u \equiv 0 \\ \hline \operatorname{supp} u \\ \hline u \equiv 0 \end{array}$$

2) $u(x, t) = a(t)$, a is any function from $(0, \infty)$ to $\mathbb{R}^3$
 $a(0) = 0.$

$$\int_0^\infty \int_{\mathbb{R}^n} a(t) (\varphi_t - \Delta \varphi) dx dt = 0 \quad \text{because:}$$

Lemma $\varphi \in D(\mathbb{R}^n, \mathbb{R}^n), \operatorname{div} \varphi = 0$ then $\int \varphi dx = 0$

Question:

How can we improve the initial condition on the weak formulation (*)? Answered! Look at the right hand side of (*).

Proof of the lemma

$$\vec{\varphi} \in D(\mathbb{R}^n, \mathbb{R}^n), \operatorname{div} \vec{\varphi} = 0 \Rightarrow \int_{\mathbb{R}^n} \vec{\varphi} dx = \vec{0}.$$

Because $\operatorname{div} \vec{\varphi} = 0$, $\int_{\mathbb{R}^n} \vec{\varphi} \cdot \nabla p dx = 0$ for all smooth function $p: \mathbb{R}^n \rightarrow \mathbb{R}$

Choose $p(x) = p(x_1, \dots, x_n) = x_1$. Then $\nabla p = (1, 0, \dots, 0)$. Then

$$0 = \int_{\mathbb{R}^n} \vec{\varphi} \cdot \nabla p dx = \int_{\mathbb{R}^n} \varphi_1 dx.$$

Similarly, $\int_{\mathbb{R}^n} \phi_i dx = 0 \quad \forall 2 \leq i \leq n$. Therefore, $\int_{\mathbb{R}^n} \vec{\phi} dx = \vec{0}$.

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Definition of solutions

$$\begin{cases} u_t - \Delta u + \nabla p = f \\ \operatorname{div} u = 0 \\ u(x, 0) = u_0(x) \end{cases} \quad (I)$$

① "Mild solution":

$$u(t) = T(t) * u_0 + \int_0^t K(t-s) * f(s) ds$$

$$p(t) = \dots$$

② Classical solution:

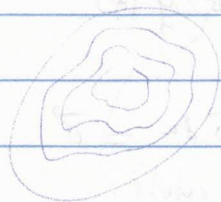
Assume f, u_0 are smooth (or Hölder continuous, ...)

u, p have derivatives & satisfy the system (I)

③ Weak solution:

$$u, p \in L^2, \dots$$

$$\varphi \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n), \operatorname{div} \varphi = 0 \Rightarrow \int_{\mathbb{R}^n} \varphi(x) dx = 0$$

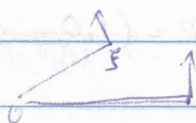


$\hat{\varphi}(\xi) =$ Fourier transform of φ

Recall: $\frac{\partial \hat{\varphi}_\alpha}{\partial x_\beta} = i \xi_\beta \hat{\varphi}_\alpha$

$$\Rightarrow \widehat{\operatorname{div} \varphi} = \sum_\alpha i \xi_\alpha \hat{\varphi}_\alpha(\xi) = 0$$

$$\Rightarrow \hat{\varphi} \perp \xi$$



$$\Rightarrow \hat{\varphi} \text{ at } 0 \text{ is } \vec{0}$$

why?

$\hat{\varphi}(0)$ if not then $\hat{\varphi}(0) \perp$

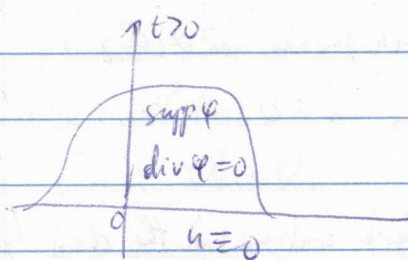
every vector in a ball centered at 0 (due to the smoothness of $\hat{\varphi}$).
This is a contradiction.

$$0 = \hat{\varphi}(0) = \int_{\mathbb{R}^n} \varphi(x) dx$$

Uniqueness of weak solutions

$$\begin{cases} u_t - \Delta u + \nabla p = f \\ \operatorname{div} u = 0 \\ u(x, 0) = u_0 \end{cases}$$

$$u \in L^1_{loc}, \quad f = u_0 = 0$$



$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^n} u(-\varphi_t - \Delta \varphi) = 0 \quad (*)$$

Example Assume $u \in L^{\infty}_{loc}$

We can assume that u is smooth. why? $u_{\varepsilon} = u * \underbrace{\phi_{\varepsilon}}_{\text{space-time mollifiers}}$

If u solves $(*)$ then u_{ε} also solves $(*)$.

$$\nabla u_{\varepsilon} = u * \nabla \phi_{\varepsilon} \lesssim \frac{1}{\varepsilon}$$

$$\nabla^2 u_{\varepsilon} = u * \nabla^2 \phi_{\varepsilon} \lesssim \frac{1}{\varepsilon^2}$$

...

$$u_{\varepsilon t} - \Delta u_{\varepsilon} + \nabla p = 0 \quad (u_{\varepsilon} \text{ is smooth, all derivatives are uniformly bounded in } \mathbb{R}^n)$$

$n=3$. $w_{\varepsilon} = \operatorname{curl}(u_{\varepsilon})$. Take curl of the equation: $w_{\varepsilon t} - \Delta w_{\varepsilon} = 0$

We know that w_{ε} is unique because of the uniqueness of heat equation. $w_{\varepsilon} = 0$ if $t \ll 0$. Thus $w_{\varepsilon} \equiv 0$

$$\Rightarrow \text{curl } u_\varepsilon(x,t) = 0$$

$$\Rightarrow u_\varepsilon(x,t) = \nabla h_\varepsilon(x,t)$$

$$\text{div } u_\varepsilon = 0 \Rightarrow \Delta_x h_\varepsilon(x,t) = 0$$

$$\Rightarrow \Delta_x u_\varepsilon(x,t) = 0$$

Liouville's Theorem: A bounded harmonic function in $\mathbb{R}^n$ is constant.

$$\Rightarrow u_\varepsilon(x,t) = a_\varepsilon(t)$$

$$\Rightarrow u(x,t) = a(t)$$

$\Rightarrow a(t)$ is the only obstacle to uniqueness in L^∞_{xt} .

If $u \in L_t^q L_x^p$ for some $p < \infty$ then $a \equiv 0$.

Corollary

Let $p < \infty$. Then weak solutions are unique in the class $L_t^q L_x^p$.

$$u = a(t), \quad p = -a'(t) \cdot x$$

If $u \in L^\infty_{xt}$ and $p \in L^\infty_{xt}$ then we have uniqueness.

$u = a(t)$ $p = -a'(t) \cdot x + r(t)$
--

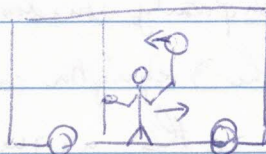
motion of container $V(t) = a(t)$

The liquid moves like a rigid body

02/17/2014

$$u(x,t) = a(t)$$

$$p(x,t) = -\frac{a'(t)}{g} x$$



when the bus brakes, the balloon goes to the left (unlike the person)

$$u_t + \Delta p = 0$$

Local weak solution of the linear Stokes-system.

$$u \in L^1_{loc}(\partial) \quad , \quad \partial \subset \mathbb{R}^n \times (t_1, t_2).$$

$$\int u (-\varphi_t - \Delta \varphi) dx dt = 0 \quad , \quad \forall \varphi \in \mathcal{D}(O, \mathbb{R}^n), \operatorname{div} \varphi = 0.$$

For heat equation: u smooth (Weyl-type theorem)

$$\text{For Stokes: } u = \text{smooth} + \underbrace{\nabla h(x, t)}_{(*)} \quad (*)$$

$$\begin{cases} \Delta_x h(x, t) = 0 \quad \forall t, \\ \text{arbitrary dependence on } t. \end{cases}$$

$$\begin{cases} u = \nabla h(x, t) \\ p = -h_t(x, t) \end{cases} \Rightarrow u_t + \nabla p = 0 \quad (\text{solves Stokes})$$

$$\begin{cases} u = \nabla h(x, t) \\ p = -h_t(x, t) - \frac{1}{2} |\nabla h(x, t)|^2 \end{cases} \quad \text{solves Navier-Stokes.}$$

balloon ^{deforms this part}
containing liquid

The fluid moves instantaneously with the boundary. If the boundary stops, the fluid immediately stops (ideal fluid).

Sketch of proof of (*)

$$w = \operatorname{curl} u \quad (\text{distribution}), \quad \operatorname{div} u = 0$$

$$w_t - \Delta w = 0$$

$$n=3: \quad \underbrace{u = \operatorname{curl} \varphi}_{\text{divergence-free}}$$

divergence-free

$$\int u (-\operatorname{curl} \varphi_t - \operatorname{curl} \Delta \varphi) = 0$$

1. w smooth

$$2. \exists \text{ some smooth } v = v(x, t), \quad \begin{cases} \operatorname{curl} v = w \\ \operatorname{div} v = 0 \end{cases}$$

$$u = v + w$$

$$\begin{cases} \operatorname{curl} w = 0 \\ \operatorname{div} w = 0 \end{cases} \Rightarrow w = \nabla h \text{ with } h \text{ harmonic.}$$

Energy inequality

1) Heat equation:

$$\begin{cases} u_t - \Delta u = \operatorname{div} f & (1) \\ u(x, 0) = u_0(x) \end{cases}$$

Multiplying both sides of (1) by u .

$$\partial_t \left(\frac{1}{2} |u|^2 \right) - (\Delta u) u = (\operatorname{div} f) u$$

$$\Rightarrow \partial_t \left(\int \frac{1}{2} |u|^2 \right) + \int |\nabla u|^2 dx \leq \int \frac{1}{2} |f|^2 + \int \frac{1}{2} |\nabla u|^2$$

$$\begin{aligned} \Rightarrow \int_{\mathbb{R}^n} \frac{1}{2} |u(x, t)|^2 dx + \int_0^t \int_{\mathbb{R}^n} \frac{1}{2} |\nabla u(x, t')|^2 dt' \\ \leq \int_{\mathbb{R}^n} |u(x, t)|^2 dx + \int_0^t \int_{\mathbb{R}^n} |f|^2 dx dt \end{aligned} \quad (2)$$

$$\text{Put } \|u\|^2 = \sup_{t \in [0, T]} \int_{\mathbb{R}^n} |u(x, t)|^2 dx + \int_0^T \int_{\mathbb{R}^n} |\nabla u(x, t)|^2 dx dt$$

$$(2) \Rightarrow \|u\|^2 \leq \|u_0\|_{L^2}^2 + \|f\|_{L_t^2 L_x^2}^2$$

2) Stokes

$$\begin{cases} u_t - \Delta u + \nabla p = \operatorname{div} f \\ u(x, 0) = u_0(x) \\ \operatorname{div} u = 0 \end{cases}$$

we still have (2) because $\int \nabla p \cdot u \, dx = 0$.

3) Navier-Stokes

$$u_t + (u \cdot \nabla) u + \nabla p - \Delta u = \operatorname{div} f, \quad \operatorname{div} u = 0$$

Note that $\int \nabla p \cdot u \, dx = 0$ and

$$\int [(u \cdot \nabla) u] \cdot u \, dx = \int u_j u_{ij} u_i \, dx = \int \underbrace{u_j}_{\text{divergence-free}} \underbrace{\left(\frac{1}{2} u_i u_i \right)_{,j}}_{\text{gradient}} \, dx = 0$$

For nondivergence form: try to write

$$n=3: \omega = \operatorname{curl} u$$

$$f = \operatorname{div} \tilde{f} \quad (??)$$

Lemma 2.4, p. 32, Temam,

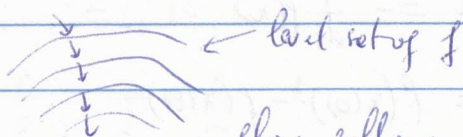
"NSE", 1977

$$\text{NSE: } u_t + \omega \wedge u + \nabla(p + \frac{1}{2} |u|^2) - \Delta u = 0$$

Other estimates

Heat equation, Stokes, are "gradient flows".

In finite dimension, $\dot{x} = -\nabla f$, $f: \mathbb{R}^n \rightarrow \mathbb{R}$.



flow follows steepest descent.

$f: X \rightarrow \mathbb{R}$, $f'(u)$ is a linear functional $\in X^*$.

$$\dot{u}(t) \in X^*$$

$$\dot{u}(t) = L f'(u) \quad L: X^* \rightarrow X.$$

Heat equation $I(u) = \int \frac{1}{2} |u|^2 \, dx$

Scalar product $(u, v) = \int u(x) v(x) \, dx$

$$\dot{u} = -\operatorname{grad} I(u)$$

$$\text{or } \frac{\partial u}{\partial t} = \Delta u$$

Stokes the same, except $X = \{u : \operatorname{div} u = 0\}$.

NSE $\dot{u} = \underbrace{\text{grad. flow}}_{\text{easy to understand}} + \underbrace{\text{Hamiltonian flow}}_{\text{(Euler flow for example) more tricky}}$

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$$-\Delta u = \text{div } f$$

$$\Leftrightarrow -\text{div } \nabla u = \text{div } f$$

The function that we want to minimize is $\int \frac{1}{2} |\nabla u|^2 - f \cdot \nabla u \rightarrow u \in H^1$

$$-\Delta u = f$$

$$\rightarrow I = \int \frac{1}{2} |\nabla u|^2 - f u \rightarrow \text{we have to leave } L^2 \text{ to show the boundedness of } I.$$

$$\dot{x} = -\nabla f$$

$$\Rightarrow |\dot{x}|^2 = -\nabla f \cdot \dot{x} = -\frac{d}{dt} f(x)$$

$$\Rightarrow \int_0^t |\dot{x}|^2 dt' = f(x(0)) - f(x(t)).$$

Typical case:
$$\begin{cases} I(u) = \int \frac{1}{2} |\nabla u|^2 \\ u_t = -\nabla I(u) \end{cases}$$

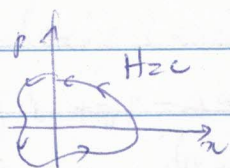
$$\frac{d}{dt} \int \frac{1}{2} |\nabla u|^2 = - \int |\nabla u_t|^2 = \int -(\Delta u)^2$$

Euler
$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = 0 \\ \text{div } u = 0 \end{cases} \quad (\text{inviscid fluid})$$

"Hamiltonian system"

$$\dim < \infty, \quad x^1, \dots, x^n, p_1, \dots, p_n$$

$$H = H(x, p) \text{ energy, } \quad \dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}$$



The solution prefers periodic behavior (instead of reaching equilibrium).

Function spaces for the linear problem $\rightarrow$ energy-related spaces

$$\begin{cases} u_t - \Delta u = \text{div} f, & f \in L^2_{tx} \\ u(x,0) = u_0 \in L^2_x \end{cases}$$

$$\|u\|^2 = \sup_t \frac{1}{2} \|u(t)\|_{L^2_x}^2 + \int_0^\infty \int_{\mathbb{R}^n} |\nabla u|^2 dx dt$$

$$\|u\| \leq C (\|u_0\|_{L^2_x} + \|f\|_{L^2_{tx}}). \text{ Question: } u, u_j \in ?$$

$$\textcircled{1} u_0 \equiv 0, \quad u(t) = \int_0^t K'(t-s) * f(s) ds$$

$\downarrow \in L_t^q L_x^p$
 $\rightarrow \in L^2_{xt}$

$$\frac{2}{q} + \frac{n}{p} = \frac{2}{2} + \frac{n}{2} - 1$$

$$n=3, p=q, u \in L^{10/3}_{tx}$$

$$n=2, p=q, u \in L^4_{tx}$$

$$n \text{ general; } p=q, u \in L^{\frac{2(n+2)}{n}}_{tx}$$

$$\|u\|_{L^{\frac{2(n+2)}{n}}_{tx}} \leq C \|f\|_{L^2_{tx}} \quad (u_0 \equiv 0)$$

$$\textcircled{2} u_0 \neq 0, f \equiv 0$$

$$\begin{array}{c} t \uparrow \\ u \in L^p_{tx} ? \\ \downarrow \\ u_0 \in L^p_x \end{array}$$

$$u_t - \Delta u = 0$$

Scaling: $u(x,t) \rightarrow \lambda u(\lambda x, \lambda^2 t)$
 $u_0(x) \rightarrow \lambda u_0(\lambda x)$

$$\Rightarrow p = \frac{2(n+2)}{n}$$

Prove that $\|u\|_{L^{p(n)}_{t,x}} \leq C \|u_0\|_{L^2_x}^2$

$$\|u\|^2 \leq \frac{1}{2} \|u\|_0^2$$

Recall: $u: \mathbb{R}^n \rightarrow \mathbb{R}$, $\|v\|_{2^*} \leq C \|v\|_2$, $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{n}$

$$\|u(t)\|_{L^p_x}^q \leq \|u(t)\|_{L^2_x}^{q(1-\alpha)} \|v u(t)\|_{L^2_x}^{q\alpha} \quad \frac{1}{p} = \frac{\alpha}{2^*} + \frac{1-\alpha}{2}$$

Choose $q\alpha = 2$.

Conclusion

$$\begin{array}{c} \uparrow \\ K * f \\ K' * f \\ \hline u_0 \in L^2 \end{array} \quad \begin{array}{c} u \in L^p_{t,x} \end{array}$$

Now $u_0 \in L^b$:

$$\begin{array}{c} \uparrow \\ u \in L^p_{t,x} \\ \hline u_0 \in L^b \end{array} \quad \text{scaling: } p = b \frac{n+2}{n}$$

Theorem $b \in (1, \infty)$, estimate is OK.

$$\|u\|_{L^{p(n)}_{t,x}} \leq \|u_0\|_{L^b_x}$$

$$\begin{array}{c} \uparrow \text{locally well posedness} \\ \text{for NSE} \\ \hline u_0 \in X \end{array}$$

$H^{1/2} C^3_x CBMO^1$: same scaling

heat extension: $U = \Gamma * u$

$$\begin{array}{c} \uparrow \\ \hline u_0 \in \mathcal{D}(\mathbb{R}^n) \end{array}$$

$$\|u\|_{X_p} := \|\Gamma u\|_{L^p_{t,x}}$$

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Nonlinear Cauchy problem:

$$\begin{cases} u_t + \nabla p - \Delta u = -\operatorname{div}(u \otimes u) \\ \operatorname{div} u = 0 \\ u(x, 0) = u_0 \end{cases}$$

$$a_i, b_j, (a \otimes b)_{ij} = a_i b_j, F = F_{ij} \\ (\operatorname{div} F)_i = \frac{\partial F_{ij}}{\partial x_j}$$

$$u(t) = \underbrace{T(t)}_{\Gamma(t)} * u_0 + \int_0^t K(t-s) * (u(s) \otimes u(s)) ds$$

$$\Rightarrow u = T + B(u, u)$$

$u \in \mathcal{X}_T$ (Banach space) suitable space of functions on $\mathbb{R}^n \times (0, T)$

This is the same idea as ODE

$$\begin{cases} \dot{x}(t) = f(x(t)) \\ x(0) = a \end{cases} \Rightarrow x(t) = a + \int_0^t f(x(s)) ds = \Phi(x)$$

Lemma X is a Banach space

$B: X \times X \rightarrow X$ bilinear, continuous

$$(\Leftrightarrow \|B(x, y)\| \leq C \|x\| \|y\|)$$

Consider $a \in X$ and the equation $x = a + B(x, x)$. (*)

If $4C\|a\| < 1$ then (*) is solvable by iteration in a ball

$$B_R = \{x: \|x\| < R\}, \text{ with } R = \frac{1 + \sqrt{1 - 4\|a\|C}}{2C}$$

$$x_0 \in B_R$$

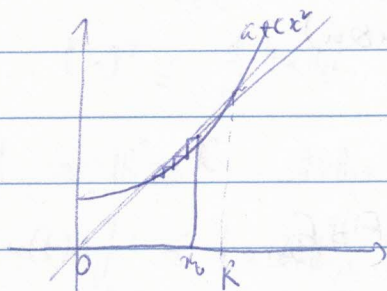
$$x_1 = a + B(x_0, x_0)$$

$$x_2 = a + B(x_1, x_1)$$



converges to a solution

when $X = \mathbb{R}$, $x = a + cx^2$



$$f(x) = a + B(x, x)$$

$$1) \|f(x) - f(y)\| < \theta \|x - y\|, \quad 0 < \theta < 1 \quad \text{in } B_R$$

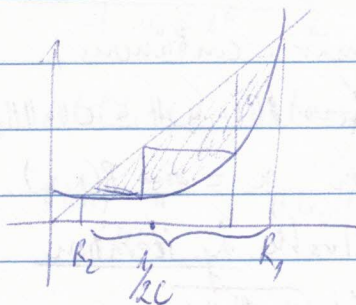
$$2) \quad x_0, \quad x_1 = f(x_0),$$

Proof of the lemma

$$\|f(x)\| \leq \|a\| + \|B(x, x)\| \leq \|a\| + C\|x\|^2$$

$$\alpha = \|a\|, \quad \xi = \|x\| \Rightarrow \|f(x)\| \leq \alpha + C\xi^2$$

$$\alpha + C\xi^2 \leq \xi \Leftrightarrow \xi \leq \frac{1 + \sqrt{1 - 4C\alpha}}{2C}$$



Check the contraction property:

$$\begin{aligned} \|f(x) - f(y)\| &= \|B(x-y, x) + B(y, x-y)\| \\ &\leq C\|x-y\|(\|x\| + \|y\|) \\ &\quad \underbrace{\qquad\qquad\qquad}_{< 2R} \end{aligned}$$

$$< 2RC\|x-y\|$$

$$\text{Want: } 2RC < 1 \Leftrightarrow R < \frac{1}{2C}$$

As long as $R < R_1$ (maybe $> 1/(2C)$), after a finite steps, the iteration method brings us to a position $< \frac{1}{2C}$

Example 1

$$\mathcal{X}_T = L_{t,x}^\infty(\mathbb{R}^n \times (0, T))$$

This is good for local in time existence when $u_0 \in L^\infty$.

ODE $\dot{x}(t) = f(x(t)) \rightarrow$ gives us only local solution.

*Philosophy:

1) local existence: for $u_0 \in X$.

2) Conserved quantities (energy): to prevent solution from blowing up ($\Rightarrow$ local solution has to be global)

For 2d, we have another conserved quantity (vorticity)

$$u(t) = \underbrace{P(t) * u_0}_U + \int_0^t -K'(t-s) * (u(s) \otimes u(s)) ds \quad (1)$$

$$\|U\|_{L^\infty_{xt}} \leq \|u_0\|_{L^\infty_x}, \quad \|K'(t-s) * f(s)\|_{L^\infty_x} \leq \|f\|_{L^\infty_x} \|K'(t-s)\|_{L^1_x}$$

$$K'(t) \sim \frac{1}{t^{\frac{n+2}{2}}} F\left(\frac{x}{\sqrt{t}}\right) = \frac{1}{\sqrt{t}} \left(\frac{1}{t^{\frac{n}{2}}} F\left(\frac{x}{\sqrt{t}}\right) \right)$$

has finite, constant L^1 -norm

$$\Rightarrow \|K'(t-s)\|_{L^1_x} \leq \frac{C}{\sqrt{t-s}}$$

$$(1) \Rightarrow \|u\|_{L^\infty_{tx}} \leq \|u_0\|_{L^\infty_x} + C \|u\|_{L^\infty_{tx}}^2 \sqrt{T}$$

On X_T : $\|B(u,v)\| \leq C\sqrt{T} \|u\| \|v\|$. Condition for solvability from the abstract lemma: $4C\sqrt{T} \|u_0\|_{L^\infty_x} \leq 1$.

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$$\begin{aligned} u &= U + \underbrace{B(u,u)} \\ &= \int_0^t -K'(t-s) * (u(s) \otimes u(s)) ds \end{aligned}$$

$$X_T = L^\infty(\mathbb{R}^n \times (0,T)), \quad u_0 \in L^\infty$$

$$(1) \|B(u, v)\|_{X_T} \leq C\sqrt{T} \|u\|_{X_T} \|v\|_{X_T}$$

$\underbrace{X_T}_{L^\infty \text{ norm on } \mathbb{R}^n \times (0, T]}$

The equation $u = U + B(u, u)$ is solvable by "Picard iteration" if $4\|u\|_{L^\infty} C\sqrt{T} < 1$. The solution is unique in B_R with

$$R \sim \frac{1}{2C\sqrt{T}}$$

Some remarks 1) $\begin{cases} u_t - \Delta u = 0 \\ u(x, 0) = u_0 \end{cases} \Rightarrow u(t) = T(t) * u_0$

The map $u: [0, T] \rightarrow L^\infty$ may not be continuous.

$$\forall t > 0, u(t) \in C_x$$

$t_j \rightarrow 0^+ \not\Rightarrow u(t_j) \rightarrow u_0$ in L^∞_x (unless u_0 is uniformly continuous).

In other words, $u \notin C([0, T], L^\infty_x)$.

2) For NSE, $u = \underbrace{T(t) * u_0}_{\text{regular}} + B(u, u)$

$$B(u, u) = - \int_0^t K'(t-s) * \underbrace{(u(s) \otimes u(s))}_{f(s) \in L^\infty_{x,t}} ds$$

$$u_t - \Delta u = \text{div} \quad \begin{array}{ccc} & L^p & \tilde{C}^{\alpha, \beta}_{par} \\ & 0 & \text{ntz} \end{array}$$

$$\Rightarrow B(u, u) \in \tilde{C}^{\alpha, \beta}_{par}$$

$$u = \underbrace{U}_{\in \tilde{C}^{\alpha, \beta}_{par}} + \underbrace{B(u, u)}_{\in \tilde{C}^{\alpha, \beta}_{par}([0, T] \times \mathbb{R}^n)}$$

may have regularity

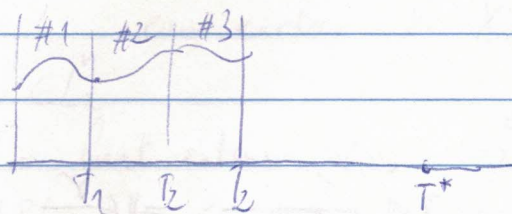
issues at $t=0$.

$\Rightarrow$ the irregularity of u at $t=0$ is caused only by the initial data.

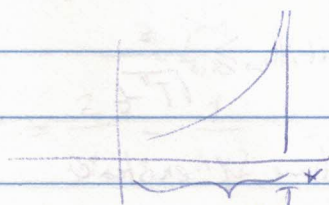
$$\begin{array}{c} \text{---} T \\ u \in C^\infty \\ \text{---} 0 \\ u_0 \end{array} \quad \text{bootstrapping regularity}$$

$$\begin{cases} \dot{x} = f(x(t)) \\ x(t) = x + \int_0^t f(x(s)) ds. \end{cases}$$

The smallness of T depends on the given u_0 . Continuation method

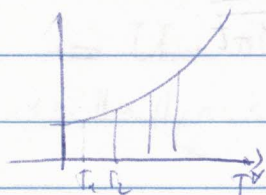


$$\dot{x}(t) = x^2(t), \quad |x(0)| = 1. \quad x(t) = \frac{1}{1-t}$$



T^* intrinsic

maximal interval of solution.



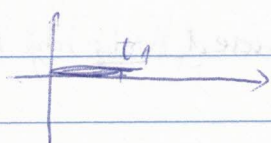
Iteration of uniqueness $\Rightarrow$ uniqueness on $(0, T^*)$.

Another setting

$$u_0 \in L_x^n, \quad \mathcal{X}_T = L_{t,x}^{n+2}(\mathbb{R}^n \times (0, T))$$

$$u = U + B(u, u) \quad \text{Similar conclusion}$$

$u_0 \in L_x^\infty \cap L_x^n$: the maximal interval of existence should be the same when we regard $u_0 \in L_x^\infty$ and $u_0 \in L_x^n$



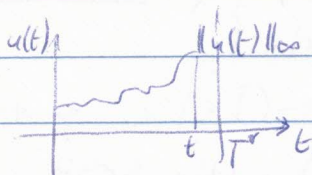
"gluing procedure"

$$u(t) = \Gamma(t) * u_0 + \int_0^t -K'(t-s) * (u(s) \otimes u(s)) ds$$

$$= \Gamma(t-t_1) * u_0(t_1) + \int_{t_1}^t -K'(t-s) * (u(s) \otimes u(s)) ds$$

$$\Gamma(s_1) * \Gamma(s_2) = \Gamma(s_1 + s_2)$$

$$\Gamma(s) * K'(t) = K'(t+s)$$



$$C \|u_0\| \sqrt{T} < 1 \Leftrightarrow T < \frac{1}{C^2 \|u_0\|_{infty}^2} = \frac{\gamma}{\|u_0\|_{infty}^2}$$

$$\frac{\gamma}{\|u(t)\|_{infty}^2} < T^* - t \Rightarrow \|u(t)\|_{infty} \geq \frac{\varepsilon}{\sqrt{T^* - t}}$$

Leray: $\exists \varepsilon > 0$, T^* maximal time of existence

$$\rightarrow \sup_{t < T^*} \sup |u(t)| \geq \frac{\varepsilon}{\sqrt{T^* - t}}$$

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$$u = U + B(u, u), \quad u, v \in L_x^\alpha = X$$

$$\|B(u, v)\|_{X_T} \leq C(T) \|u\|_{X_T} \|v\|_{X_T}$$

"subcritical spaces", "subcritical theory"

1) $u \in X_T \Rightarrow B(u, u)$ higher regularity (transparently)

$$2) \lambda u(\lambda x, \lambda^2 t) = u_\lambda(x, t)$$

$$\lambda u_0(\lambda x) = u_{0\lambda}(x)$$

$$\|u_{0\lambda}\|_X \rightarrow \infty \text{ as } \lambda \rightarrow \infty$$

$$3) \|B(u, u)\|_{X_T} \leq C(T) \|u\|_{X_T} \|u\|_{X_T} \text{ (depends on } T)$$

4) Smallness of $\|u_0\|_X$ ("relative" due to scaling)

$\sqrt{T} \|u_0\|_X$ is scale-invariant

Ex: $X = L_x^p$, $p > n$

$X_T = C(p, T, L_x^p)$

Critical theory

$X = L_x^n$, $X_T = L_{t,x}^{n+2}(\mathbb{R}^n \times (0, T))$

(Kato, 1980s: $X = L_x^n$ but with different X_T)
even earlier with $X = H^{1/2}$

1) $u_0 \in L_x^n$

U - heat extension of $u_0 \Rightarrow \|U\|_{L_{t,x}^{n+2}} \leq C \|u_0\|_{L_x^n}$

2) $\|B(u, v)\|_{X_T} \leq C \|u\|_{X_T} \|v\|_{X_T}$

$w_t - \Delta w + \nabla q = - \frac{2}{\partial x_j} (u v)$
 $\downarrow \in L_{t,x}^{n+2}$ $\in L_{t,x}^{\frac{n+2}{2}}$

$\frac{1}{p} = \frac{2}{n+2} - \frac{1}{n+2} = \frac{1}{n+2}$

C doesn't depend on T

$u = U + B(u, u)$ solvable if $4 \|U\|_{X_T} C \leq 1$

① $\|u_0\|_{L_x^n} < \varepsilon_0$ global existence

$4 \|U\|_{X_T} C \rightarrow 0$ as $T \rightarrow 0$

$\Rightarrow$ if $\|u_0\|_{L_x^n} < \infty$ then $\exists$ local solutions

$u_{tt} = U + B(u, u)$

how to improve the inequality of this R term

critical $\Rightarrow$ cannot lift L^{n+2} to $L^{n+2+\varepsilon}$
spaces

$$u \in L^{n+2}_{t,x} \Rightarrow B(u, u) \in C_t L^n_x$$

Linear estimate

$$\begin{cases} u_t - \Delta u = \text{div } f \\ u(x, 0) = 0 \end{cases}$$

$$f \in L^{n+2}_{t,x}. \text{ Then } u \in C_t L^n_x$$

For $n=2$: $f \in L^2_{t,x}$

$$u \in [0, T] \rightarrow L^n_x \text{ continuous.}$$

$$\frac{\partial}{\partial t} (u(t), u(t))_{L^n_x} = 2(u(t), u'(t))_{L^n_x} = 2(u(t), \Delta u(t) + \text{div } f)$$

$$= \int_{\mathbb{R}^n} -2|\nabla u(x, t)|^2 dx - \int_{\mathbb{R}^n} \nabla u \cdot f dx \in L^1$$

$\Rightarrow (u(t), u(t))_{L^n_x}$ is absolutely continuous.

"Duality method"

$$\begin{array}{c} \uparrow \\ \text{--- } u_t(x) \text{ --- } t \\ \downarrow \\ v_t + \Delta v = 0 \\ \\ \text{--- } u_t - \Delta u = \text{div } f \text{ ---} \\ \downarrow \\ u(x, 0) = 0 \end{array} \quad \int u(x, t) v_t(x) dx$$

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^n} u(x, t) v(x, t) dx = \int u_t v + u v_t = \int (\Delta u + \text{div } f) v + u (-\Delta v) dx$$

$$= \int -\nabla u \cdot \nabla v$$

$$\Rightarrow \int \frac{\partial}{\partial t} (u(x, t) v(x, t)) dx = \int_0^t \int_{\mathbb{R}^n} -\nabla u \cdot \nabla v dx dt$$

no dependence on u

Example $f=0 \Rightarrow u=0$ ($u_t - \Delta u = f$)

$$f=0 \Rightarrow \int v(x) u(x, t) dx = 0 \quad \forall v \in D(\mathbb{R}^n) \Rightarrow u=0.$$

Thus, if the integration-by-part formula is valid then we have the uniqueness. This is true when u grows not too fast as $|x| \rightarrow \infty$. By the explicit formula of v , $|v| \leq e^{-r|x|^2}$. If $u \leq e^{C|x|^2}$ for some C then we have the uniqueness.

"Duality method" because instead of studying the growth of u , we study the growth of v .

Question: For critical theory, $u_0 \in L^p_x$, $u \in L^{n+2}_{loc}$, how can we retrieve the initial condition? because we cannot say $u(x,0)$ (not well-defined). In what sense should we understand the initial condition?

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$$u = U + B(u, u)$$

$$\|u\|_{L^n_x} < \infty \Rightarrow \exists \text{ local solution } u \in C_t L^n_x.$$

$$-\Delta u = u^3, \quad n=5 : \text{analogous to 3D NSE.}$$

$$n=4 : \text{analogous to 2D NSE.}$$

$$u \in L^p_{loc}, \quad p > 5 \rightarrow \text{"subcritical"} \rightarrow \text{smooth.}$$

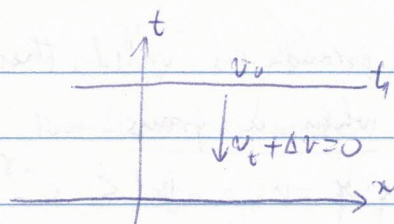
Example: $u(x) = \frac{A}{|x|}$ is a solution (suitable A).

Question: $\left. \begin{array}{l} -\Delta u = u^3 \\ u \in L^5_{loc} \end{array} \right\} \Rightarrow \text{smoothness?} \quad \text{YES! but the proof is subtle.}$

(proved in 1960s)

$$\begin{cases} u - \Delta u + \nabla p = \text{div } f & \text{if } f \in L^{\frac{n+2}{2}}_{t,x} \\ \text{div } u = 0 \\ u(x,0) = 0 \end{cases}$$

want to show that $u \in C_t L^n_x$. Idea: duality method.



$$\int_{\mathbb{R}^n} u(x, t_1) v_0(x) dx \leq C \|v_0\|_{L_x^{\frac{n}{n-1}}} \quad (1)$$

$$\forall v_0 \in L_x^{\frac{n}{n-1}}$$

Can assume $\text{div } v = 0$ because of the Helmholtz decomposition.

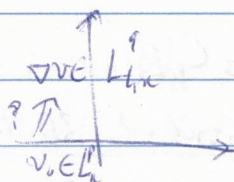
$$(v = P v + \nabla \phi)$$

$$\text{LHS}(1) = \int_0^{t_1} \int_{\mathbb{R}^n} -\nabla v \cdot f \, dx dt$$

$f \in L_{t,x}^{\frac{n+2}{2}}$
 $\in L_{t,x}^{(\frac{n+2}{2})'}$

want:

$$\|\nabla v\|_{L_{t,x}^{(\frac{n+2}{2})'}} \leq C \|v_0\|_{L_x^{\frac{n}{n-2}}}$$



Scaling argument: $v \mapsto v(\lambda x, \lambda^2 t)$

$$q = \frac{(n+2)p}{p+n}$$

$$p=2: q=2$$

$$p=1: q = \frac{n+2}{n+1}$$

} Morikawitz Interpolation

$$\int |\nabla v(x, t)|^{\frac{n+2}{n-2}} dx = \frac{n+2}{2} \quad \text{border line}$$

$$v_0 \in L^1 \not\Rightarrow \nabla v \in L_{t,x}^{\frac{n+2}{n-1}}$$

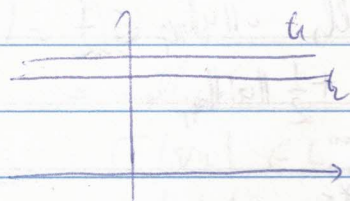
$$\Rightarrow \nabla v \in L_{t,x}^{\frac{n+2}{n-1}, \infty}$$

weak space.

$$\|\nabla v\|_{L_{t,x}^{(\frac{n+2}{2})'}} \leq C \|v_0\|_{L_x^{\frac{n}{n-2}}} \rightarrow \text{ok.}$$

$$\rightarrow u \in L_t^\infty L_x^n$$

How to show $u \in C_t L_x^n$?



$$\int [u(x, t_1) - u(x, t_2)] \varphi(x) dx$$

$$= u(x, t_1 + \tau) - u(x, t_2)$$

$$[u(x, t + \tau) - u(x, t)] \varphi(x) = \Delta(\dots)$$

$$u(x, t + \tau) + \nabla(\dots) = \text{div}(\tilde{f}(x, t + \tau) - \tilde{f}(x, t))$$

————— t_1

$$\frac{\int_{t=0}^{t_1} u}{\varphi=0} = \int -\nabla v [\tilde{f}(x, t + \tau) - \tilde{f}(x, t)] dx dt$$

$$u=0 \quad \left\{ \begin{array}{l} t < 0 \\ \|\cdot\|_{L_{t,x}^{n+2}} \rightarrow 0 \end{array} \right.$$

Maximal time of existence

$$\|u\|_{L_{t,x}^{n+2}(\mathbb{R}^n \times (0, t^*))} = \infty$$

Ladyzhenskaya - Prodi - Serrin

$$n=2: \quad n+2=4$$

$$\text{energy estimate} \Rightarrow \|u\|_{L_{t,x}^4} < \infty \Rightarrow \exists \text{ global solution.}$$

05/03/2014

Regularity $u = \underbrace{U}_{\tilde{U}(t) * u_0} + B(u, u) \quad (1)$

i) Propagation of regularity

- Subcritical, $X_T = L_{x,t}^\infty(\mathbb{R}^n \times (0, T))$, $u_0 \in L_x^\infty$, $\nabla u_0 \in L_x^\infty$

$$\Rightarrow U \in L_{x,t}^\infty, \quad \nabla_x U = L_{t,x}^\infty$$

Put $u' = \frac{\partial u}{\partial x_j}$ Take derivative of (1).

$$u' = U' + B(u', u) + B(u, u') = U' + \tilde{B}(u, u')$$

Thus, $\|u'\|_{X_T} \leq \|U'\|_{X_T} + \tilde{c} \|u\|_{X_T} \|u'\|_{X_T}$ (2)

depend on T

If T is small or u_0 is small, $\|u\|_{X_T} \leq \frac{1}{2}$ Then

$$(2) \Rightarrow \|u'\|_{X_T} \leq \|U'\|_{X_T} + \frac{1}{2} \|u'\|_{X_T}$$

$$\Rightarrow \frac{1}{2} \|u'\|_{X_T} \leq \|U'\|_{X_T}$$



If $\|u\|_{X_T} < \infty$ then $u' \in X_T$ by following step by step of the

Conclusion: $u' \in X_T \forall T < T^*$, the max. time interval of u .

Technicalities

need to know: $\|u'\|_{X_T} < \infty$

need an estimate when $\tilde{c} \|u\|_{X_T} < \frac{1}{2}$.

Remove: work with $\frac{u(x+ch) - u(x)}{h} = D_1^h u$

$$\Rightarrow D_1^h \tilde{B}(u, u) = \tilde{B}(D_1^h u, u) + \tilde{B}(u, D_1^h u)$$

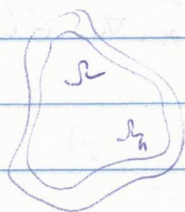
"discrete Leibniz rule"

$$\Rightarrow \text{obtain } \|D_1^h u\|_{X_T} \leq 2 \|D_1^h U\|_{X_T} \quad (\tilde{c} \|u\|_{X_T} < \frac{1}{2})$$

$+ h \in (0, h_0)$

same lemma $\Rightarrow \|u'\|_{X_T} \leq 2 \|U'\|_{X_T}$

Lemma



$u: \Omega \rightarrow \mathbb{R}^m$, $\Omega \subset \mathbb{R}^n$. Then it is equivalent:

(1) $\partial_i u$ (distrib. derivative) $\in L^p(\Omega)$, $1 \leq p \leq \infty$.

(2) $\exists h_0 > 0$, $\int_{\Omega_h} |\partial_i^h u|^p dx \leq C \quad \forall h \in (0, h_0)$

Improved version: smoothing of u_0

(I) Heat equation, $u_t - \Delta u = 0$, $u(x, 0) = u_0 \in L^\infty$

$$u(t) = \underbrace{\Gamma(t)}_{\approx \text{multiplier}} * u_0$$

$\approx \text{multiplier}$

Lemma: $\|\Gamma \nabla u\| \in L_{t,x}^\infty$

$$t \|\nabla^2 u\| \in L_{t,x}^\infty$$

$$t^{\frac{m}{2}} \|\nabla^m u\| \in L_{t,x}^\infty$$

$$\|t^{\frac{m}{2}} \nabla^m u\|_{L_{t,x}^\infty} \leq C_m \|u_0\|_{L_x^\infty}$$

$$\|t^{\frac{m}{2}+l} \partial_t^l \nabla^m u\|_{L_{t,x}^\infty} \leq C_{m,l} \|u\|_{L_x^\infty}$$

Example:

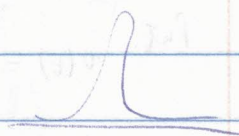
$$u(t) = \Gamma(t) * u_0$$

$$u'(t) = \Gamma'(t) * u_0 \Rightarrow \Gamma u'(t) = \Gamma \Gamma'(t) * u_0$$

$$\Gamma(t) = \frac{1}{t^{n/2}} F\left(\frac{x}{\sqrt{t}}\right)$$

$$\Rightarrow \Gamma \Gamma'(t) = \frac{1}{t^{n/2}} F'\left(\frac{x}{\sqrt{t}}\right)$$

$$\text{look like } \phi_\varepsilon = \varepsilon^{-n} \phi\left(\frac{x}{\varepsilon}\right)$$



Smoothing for $u_0 \in L_x^p$:
$$\begin{cases} u_t - \Delta u = 0 \\ u(x,0) = u_0 \end{cases}$$

$$u_0 \in L_x^p, \quad u \in C_t^\infty L_x^p, \quad t^{\frac{m}{2}+l} \partial_t^l \nabla^m u \in L_t^\infty L_x^p$$

$$u \in L_{t,x}^{p, \frac{n+p}{n}}(\mathbb{R}^n \times (0, \infty)) = X$$

$$\|u\|_X \leq C_p \|u_0\|_{L_x^p}, \quad p > 1$$

$$\|\Gamma \nabla u\|_X \leq C_{p,1} \|u\|_{L_x^p}$$

$$\|t \nabla^2 u\|_X \leq \dots$$

} proof by interpolation

② Smoothing for NSE $u = U + \tilde{B}(u, u)$ (mild solutions)
viewed as a small perturbation of U

Expect that u is close to U in terms of regularity at least for a short time. Follow similar approach as the case of heat equations.

$$u' = \frac{\partial u}{\partial t} \Rightarrow u' = U' + \tilde{B}'(u, u)$$

$$\Rightarrow u'(t) = U'(t) + \underbrace{\left[\int_0^t -K'(t-s) * (u(s) \otimes u(s)) ds \right]}'_I$$

$$I = \int_0^{t/2} -K'(t-s) * (u(s) \otimes u(s)) ds + \int_{t/2}^t -K'(t-s) * (u(s) \otimes u(s)) ds$$

↑
put derivative here
↑
put derivative here

03/05/2014

Smoothing effect

$$u = U + \tilde{B}(u, u)$$

$$\text{Put } v(t) = \sqrt{t} \frac{\partial u}{\partial t} = \sqrt{t} u'$$

$$u'(t) = U'(t) + \left[\int_0^t -K'(t-s) * (u(s) \otimes u(s)) ds \right]'$$

$$= U'(t) + \int_0^{t/2} -K''(t-s) * (u(s) \otimes u(s)) ds + \int_{t/2}^t -K'(t-s) * (u'(s) \otimes u(s) + u(s) \otimes u'(s)) ds$$

Thus,

$$\sqrt{t} u'(t) = \sqrt{t} U'(t) + \underbrace{\int_0^{t/2} -\sqrt{t} K''(t-s) * (u(s) \otimes u(s)) ds}_{\{1\}} +$$

$$+ \underbrace{\int_{t/2}^t -K'(t-s) * \frac{\sqrt{t}}{\sqrt{s}} (\sqrt{s} u'(s) \otimes u(s) + u(s) \otimes \sqrt{s} u'(s)) ds}_{\{2\}}$$

Consider {1}: $K'(t-s) = \frac{1}{(t-s)^{\frac{n+1}{2}}} F\left(\frac{x}{\sqrt{t-s}}\right)$

$$\Rightarrow \sqrt{t} K''(t-s) = \underbrace{\sqrt{\frac{t}{t-s}}}_{\leq \sqrt{2}} \frac{1}{(t-s)^{\frac{n}{2}}} F'\left(\frac{x}{\sqrt{t-s}}\right)$$

Consider {2}:

$$\{2\} \sim C |K'(t-s)| \times |u(s) \otimes u(s)|$$

→ same estimate for $B(u, u)$ ($\leq C \|u\|_{X_T}^2$)

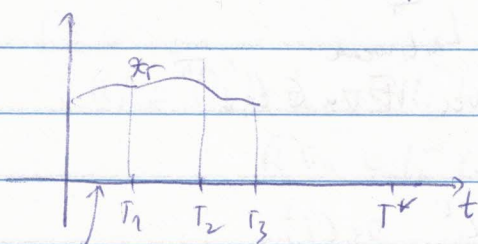
$$\{2\}: \underbrace{\sqrt{\frac{t}{s}} (-K'(t-s)) \otimes [v(s) \otimes u(s) + u(s) \otimes v(s)]}_{\leq \sqrt{2}} \xrightarrow{\text{after } \int_0^{t/2}} \underbrace{B(v, u)(t) + B(u, v)(t)}_{\| \cdot \|_{X_T} \leq C \|u\|_{X_T} \|v\|_{X_T}}$$

Put $V = \sqrt{t} U'$. Then

$$\|u\|_{X_T} \leq \|V\|_{X_T} + C \|u\|_{X_T}^2 + C \|u\|_{X_T} \|v\|_{X_T}$$

If $C \|u\|_{X_T} < \frac{1}{2}$ then

$$\|u\|_{X_T} \leq 2 \|V\|_{X_T} + 2C \|u\|_{X_T}^2$$



$$C \|u\|_{X_{T_1}} < \frac{1}{2}$$

$$u = U + B(u, u)$$

$$u'' = U'' + B(u'', u) + 2B(u', u') + B(u, u'')$$

$$t u' = w, \quad t U'' = w$$

$$\left. \begin{array}{l} t \nabla u \\ t \nabla^2 u \\ t^{3/2} \nabla^3 u \end{array} \right\} \text{ similar.}$$

$$\begin{aligned}
 u''(t) &= U''(t) + \left[\int_0^{t/2} -K'(t-s) * (u(s) \otimes u(s)) ds \right]'' + \\
 &\quad + \left[\int_{t/2}^t -K'(t-s) * (u(s) \otimes u(s)) ds \right]'' \\
 &= U''(t) + \int_0^{t/2} \underbrace{-K'''(t-s)}_{\text{is not worse than } C \frac{t}{t-s} K'(t-s)} * (u(s) \otimes u(s)) ds + \int_{t/2}^t -K'(t-s) * \underbrace{[u(s) \otimes u(s)]''}_{\frac{t}{s} u''(s) \otimes u(s)} ds
 \end{aligned}$$

is not worse than $C \frac{t}{t-s} K'(t-s)$

$$\frac{t}{s} u''(s) \otimes u(s)$$

$t^{\frac{m}{2}} \nabla^m U \in \mathcal{X}_T$, obvious for $\mathcal{X}_T = L_{t,n}^\infty$, $u_0 \in L_n^\infty$.

Consider $\mathcal{X}_T = L^{\frac{n+2}{n}}(R^n \times (0,T))$, $u_0 \in L_n^n$

$$\left. \begin{aligned} u_t - \Delta u &= 0 \\ u(x,0) &= u_0 \in L_n^n \end{aligned} \right\} u \in L_{t,n}^{p, \frac{n+2}{n}}$$

Claim: $\nabla \nabla u \in L_{t,n}^{p, \frac{n+2}{n}}$

Proof Interpolation

(i) true for $u_0 \in L_n^\infty$

(ii) true for $u_0 \in L_n^1$, and $L_{t,n}^{\frac{n+2}{n}}$ weak

Marcinkiewicz interpolation gives $\nabla \nabla u \in L_{t,n}^{p, \frac{n+2}{n}}$

The case $u_0 \in L_n^2$:

$$u_t - \Delta u = 0 \rightarrow u \in L_t^\infty L_n^2, \nabla u \in L_{t,n}^2$$

$$v = \nabla \nabla u$$

$$(\text{?}) \quad v \in L_t^\infty L_n^2, \nabla v \in L_{t,n}^2$$

$$v(t) = \underbrace{(\nabla \nabla \Gamma(t))}_{\text{"multiplier"}} * u_0$$

$$\|v(t)\|_{L^2} \leq C \|u_0\|_{L^2}$$

$$u_t \sqrt{t} \nabla^2 u \in L^2_{tx} ?$$

$$u(t, x) \xrightarrow[\text{transform}]{\text{Fourier}} \hat{u}(\xi, t)$$

$$\hat{u}_t = -|\xi|^2 \hat{u}(\xi, t)$$

$$\hat{u}(\xi, t) = \hat{u}_0(\xi) e^{-|\xi|^2 t}$$

$$\|t \nabla^2 u(t)\|_{L^2_x}^2 \approx \|t |\xi|^2 \hat{u}_0(\xi) e^{-|\xi|^2 t}\|_{L^2_x}^2$$

Note se^{-s} is bounded

Question: how can we apply F-transform if we haven't known that $u_0 \in L^2_x$?

Answer consider u_0 compactly supported, then

$$\|t \nabla^2 u\|_p \leq C \|u_0\|_p$$

then pass through limit to get the case $u_0 \in L^2_x$.

03/07/2014

Energy estimate

$$u = U + B(u, u) \quad \text{"mild solution"}$$

$$\begin{cases} u_t - \Delta u = \text{div } f \\ u(0) = u_0 \in L^2_x \end{cases}, f \in L^2_{tx}$$

1) $f = 0$.

$$u(t) = \Gamma(t) * u_0 \in L^\infty_t L^2_x$$

$$\sqrt{t} \nabla u \in L^\infty_t L^2_x$$

$$\begin{aligned} \nabla u(t) &= \Gamma'(t) * u_0, \quad \|\nabla u(t)\|_p = \|\Gamma'(t) * u_0\|_p \\ &\leq \underbrace{\|\Gamma'(t)\|_{L^1}}_{\approx \frac{C}{\sqrt{t}}} \|u_0\|_{L^2} \end{aligned}$$

$$\Rightarrow \|\nabla u(t)\|_{L^2} \leq \frac{C}{\sqrt{t}} \|u_0\|_{L^2}$$

$\Rightarrow$ cannot show that $\nabla u \in L^2_{tx}$!!

→ have to use the fact that $\int \Gamma'(t) dt = 0$ ^{dx?}

2) $u_0 \equiv 0, f \neq 0$

$$u(t) = \int_0^t \Gamma(t-s) * \operatorname{div} f(s) ds = \int_0^t \nabla \Gamma(t-s) * f(s) ds$$

$$u(t) = \int_0^t \Gamma'(t-s) * f(s) ds$$

$$\|u(t)\|_{L_x^2} \leq \int_0^t \|\Gamma'(t-s) * f(s)\|_{L_x^2} ds \leq \int_0^t \underbrace{\frac{C}{\sqrt{t-s}} \|f(s)\|_{L_x^2}}_{\text{don't know how to estimate!}} ds$$

don't know how to estimate!

~~$-\Delta u = \operatorname{div} f, f = (f_1, \dots, f_n)$~~

$-\Delta u = \operatorname{div} f, f = (f_1, \dots, f_n)$

Fourier transform: $|\xi|^2 \hat{u}(\xi) = i\xi \hat{f}(\xi)$

$$i\xi \hat{u}(\xi) = \frac{-\xi \otimes \xi \hat{f}(\xi)}{|\xi|^2}$$

bounded in L^2

$u_t - \Delta u = \operatorname{div} f$

$$(i\tau + |\xi|^2) \hat{u}(\xi, \tau) = i\xi \hat{f}(\xi)$$

$$\Rightarrow \hat{u}(\xi, \tau) = \frac{i\xi \hat{f}(\xi)}{i\tau + |\xi|^2}$$

$$\nabla u \in L_t^2 L_x^2, \quad i\xi \hat{u}(\xi, \tau) = \underbrace{\frac{-\xi \otimes \xi \hat{f}(\xi, \tau)}{i\tau + |\xi|^2}}_{\text{bounded}} \in L_{t,x}^2$$

Recall how we get the energy estimates for elliptic / heat equations.

$$-\Delta u = \operatorname{div} f \Rightarrow \int |\nabla u|^2 = \int -f \nabla u \leq (\int f^2)^{1/2} (\int |\nabla u|^2)^{1/2}$$

$$\Rightarrow \| \nabla u \|_2 \leq \| f \|_2$$

$\Rightarrow$ we get this estimate not through the simple convolution estimate!

$$u_t - \Delta u = \text{div} f$$

$$\Rightarrow \frac{\partial}{\partial t} \int \frac{|u|^2}{2} + \int |\nabla u|^2 = \int f \nabla u$$

... energy estimate

Lemma $f \in L^1_{tx}, u_0 \in L^2_x \Rightarrow u$ is in the energy class
+ satisfies the energy inequality.
"mild solution"

Navier-Stokes:

$$u = U + B(u, u) \quad \text{"mild solution"}$$

$$u, \in L^\infty_t \cap L^2_x$$

↓
perturbation theory energy class

Claim: $u_0 \in L^\infty \cap L^2$, T^* maximal time of existence of the mild solution $\Rightarrow \forall T < T^*$, $u \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$ + energy inequality is satisfied.

Proof Put $u_{n+1} = U + B(u_n, u_n)$

$$\begin{aligned} \| \underbrace{u_{n+1} - u_n}_{v_{n+1}}(t) \|_2 &\leq \| B(u_n, u_n)(t) - B(u_{n-1}, u_{n-1})(t) \|_2 \\ &= \underbrace{B(u_n - u_{n-1}, u_n)}_{\{I\}} + B(u_{n-1}, u_n - u_{n-1}) \end{aligned}$$

$$\text{Put } \sup_{t \in [0, T]} \| v_n(t) \|_2 = \gamma_n$$

$$\left\| \int_0^t -K(t-s) * (v_{n-1}(s) \otimes u_n(s)) ds \right\|_2^2$$

$$\leq 2 \| u \|_\infty \gamma_n \int_0^t \underbrace{\frac{C}{\sqrt{t-s}}}_{C\sqrt{t}} ds$$

$\gamma_{n+1} \leq \frac{1}{2} \gamma_n$ on (qT_1) , provided that $\frac{2\|u\|_{\infty} C F_1}{\gamma_1} < \frac{1}{2}$
 Then continue the method on $(t_1, T_2), (T_2, T_3), \dots$

03/10/2014

No class on Friday! Professor will leave right after class on Wednesday.
 First homework due March 26th.

$$u = U + B(u, u)$$

$$u_0 \in L_x^\infty \cap L_x^2 \Rightarrow u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1 \cap L_{tx}^\infty + \text{energy estimate.}$$

Recall regularity result: $\forall t \forall u \in L_{tx}^\infty, \dots$

Conceptually:

1) mild solution: "basic"

~~*~~ energy class: "regularity"

2) energy: "basic" (weak-solution)

regularity: "properties of mild solutions"

Critical theory: consider $u_0 \in L_x^n \cap L_t^2$

$$\text{Energy norm: } \|u\|_T^2 = \sup_{0 \leq t \leq T} \|u(t)\|_{L_x^2}^2 + \int_0^T \|\nabla u(t)\|_{L_x^2}^2 dt.$$

Parabolic embedding: $L_t^\infty L_x^2 \cap L_t^2 H_x^1 \hookrightarrow L_{tx}^{2^*p}$, where

$$\frac{1}{2^*p} = \frac{1}{2} - \frac{1}{n+2}$$

In particular,

$$\|vw\|_{L_{tx}^2(\mathbb{R}^n \times (0, T))} \leq C \|v\|_T \|w\|_{L_{tx}^{n+2}(\mathbb{R}^n \times (0, T))}$$

Recall: $u_t - \Delta u + \nabla p = -\frac{\partial}{\partial y_j} (y_j u)$

If we can control $y_j u$ in L_{tx}^2 then we can control u in $\|u\|_T$

A backward question: $\|u\|_T \xrightarrow{\text{control}} \|y_j u\|_{L_{tx}^2} \leq C \|u\|_T \|u\|_{L_{tx}^{n+2}}$

$$u_{k+1} = U + B(u_k, u_k)$$

$$\begin{aligned} v_{k+1} &= u_{k+1} - u_k = B(u_k, u_k) - B(u_k, u_{k-1}) \\ &= B(u_k, u_k - u_{k-1}) + B(u_k - u_{k-1}, u_{k-1}) \\ &= B(u_k, v_k) + B(v_k, u_{k-1}) \end{aligned}$$

$$\text{We have } v_{k+1,t} - \Delta v_{k+1} + \nabla q_{k+1} = \frac{\partial}{\partial y} (u_{k,j} v_{k-1} + v_{k,j} u_{k-1})$$

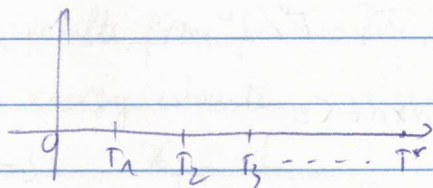
$$v_k(x, 0) = 0$$

known

$$\begin{aligned} \|v_{k+1}\|_T &\leq C \underbrace{(\|u_k\|_{L^{n+2}_{t,x}} + \|u_{k-1}\|_{L^{n+2}_{t,x}})}_{\leq \frac{1}{2} \text{ if } T=T_1 \text{ small}} \|v_k\|_T \\ &\leq \frac{1}{2} \text{ if } T=T_1 \text{ small.} \end{aligned}$$

$$\|v_{k+1}\| = \|u_{k+1} - u_k\| \leq \frac{1}{2} \|v_k\|$$

$\rightarrow u_k$ Cauchy in the energy-norm on $(0, T_1)$.



$(0, T^*)$: maximal interval of existence of mild solution

(determined by the equation $u = U + B(u, u)$ and the space X_T^1)

Question: why $T_k \rightarrow T^*$?

Smallness condition for steps from T_j to T_{j+1} : $\exists \varepsilon > 0$:

$$\|u\|_{L^{n+2}_{t,x}(\mathbb{R}^n; (T_j, T_{j+1}))} \leq \varepsilon^{-\frac{n+2}{2}}$$

$$f(t) = \int_{\mathbb{R}^n} |u(x, t)|^{n+2} dx$$

$$\Rightarrow \int_{T_j}^{T_{j+1}} f(t) dt < \varepsilon^{-\frac{n+2}{2}}$$

$$\Rightarrow \forall T < T^*, f \in L^1(0, T).$$

$$F(t) = \int_0^t f(s) ds \quad \text{absolutely continuous.}$$

$$\exists \delta > 0 \text{ s.t. } |t_1 - t_2| < \delta \Rightarrow |F(t_1) - F(t_2)| < \varepsilon^{n+2}$$



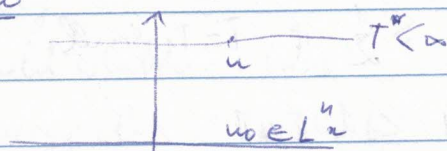
We can "march" to T with finitely many steps (each less than δ)

Recall: subcritical case

$$\|B(u, v)\|_{X_T} \leq \underbrace{C(T)}_{L^\infty_{tx} C \sqrt{T}} \|u\|_{X_T} \|v\|_{X_T}$$

smallness in this case only depends on T , not on the solution u .

Example



$$u_t - \Delta u = u^3$$

$$u_\lambda = \lambda u(\lambda x, \lambda^2 t)$$

$$\|u_\lambda\|_{L_x^n} = \|u\|_{L_x^n}, \quad T_\lambda^+ = T^+ / \lambda^2$$

$$u_0, U, \int_0^T \int_{\mathbb{R}^n} |U|^{n+2} dx dt \leq \varepsilon$$

03/12/2014

Reminders: No class on Friday.

$$u = U + B(u, u)$$

$$\|u\| < \infty, \quad u \in X_T$$

$$\frac{1}{2} \int u^2(x, t_2) dx + \iint |u(x, t)|^2 dx dt = \frac{1}{2} \int u^2(x, t_1) dx.$$

$$0 \leq t_1 \leq t_2.$$

$$u_t + u(u \cdot \nabla) u + \nabla p - \Delta u = 0 \quad (*)$$

How to recover p ?

$$-\Delta p = \operatorname{div}(u \cdot \nabla u) = \partial_i \partial_j (u_i u_j)$$

$$\text{Calderon-Zygmund: } u \in L^q \Rightarrow \exists! p \in L^{\frac{q}{2}} \quad (q > 2).$$

Critical case: $u \in L_t^\infty L_x^n$, $p \in L_t^\infty L_x^{\frac{n}{2}}$

Energy: $u \in L_t^\infty L_x^2$, $p \in L_t^\infty L_x^q$ $\forall q \in (1, \frac{n}{2}]$.

How to prove? Multiply both side of (*) by $u\varphi$, where φ is a cut-off function:

$$\partial_t \int_{\mathbb{R}^n} \frac{|u|^2}{2} \varphi + \int_{\mathbb{R}^n} |u|^2 \varphi = \int \pm \frac{u^2}{2} \Delta \varphi + \int \pm (u \cdot \nabla \varphi) \left(\frac{u^2}{2} + p \right)$$

Take $\varphi = \varphi_\varepsilon$, $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$. Let $\varepsilon \rightarrow \infty$.

(1) $n=2$: ($4=n+2$)

$$X_T = L^4(\mathbb{R}^2 \times (0, T)), \quad u_0 \in X = L_x^2$$

$$\|u\|_{2,p} \leq C\|u\|$$

$$\frac{1}{2^*} = \frac{1}{2} - \frac{1}{n+2} \Rightarrow 2^* = 4$$

$\Rightarrow$ the results from perturbation theory ~~ex~~ match exactly with those from energy estimate.

(2) $n=3$: $X_T = L_{tx}^5$ } the two theories don't match
energy $\| \cdot \|$: $L_{tx}^{\frac{10}{3}}$

Local perturbation:

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad \|u_\lambda\|_X = \|u\|_X$$

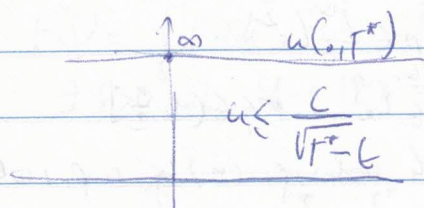
$$\|u_{\lambda^2}\|_X = \|u_0\|_X$$

$\rightarrow$ borderline case!

$$\text{Energy: } \|u_\lambda\| = \frac{1}{\lambda} \|u\|$$

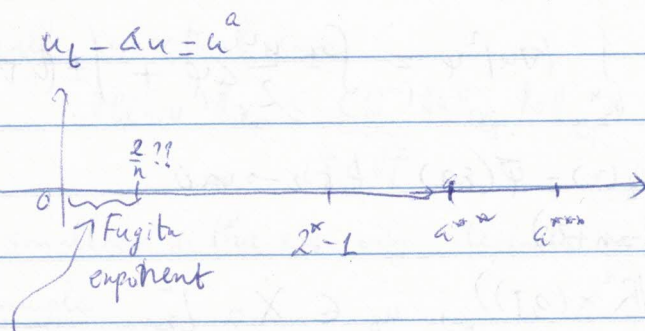
$\Rightarrow$ we get less information from the energy estimate than from the critical case.

$u_1 - \Delta u = u_3$: the solution may blow at some point, even with compactly-supported initial data.



But the energy remains finite thus the energy doesn't control the blowing up.

$$\|u\|_{T^*} < \infty$$



all solutions blow up.

CGL (complex Ginzburg-Landau equation)

$$u: \mathbb{R}^3 \times (t_1, t_2) \rightarrow \mathbb{C}, \quad i u_t + (1 - i\varepsilon) \Delta u + |u|^2 u = 0 \quad (\varepsilon > 0)$$

$$\int \frac{1}{2} |u(t_1, t_2)|^2 + \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \varepsilon |u|^2 dx dt = \int \frac{1}{2} |u(t_1)|^2$$

Scaling: $u(x, t) \rightarrow \lambda u(\lambda x, \lambda^2 t)$ (very similar to NSE)

For 3D: this equation blows up.

Burger's equation

$$u_t + (u \cdot \nabla) u - \Delta u = 0$$

$$\text{NSE: } u_t - \Delta u + \nabla p = - \underbrace{\nabla \cdot (u \otimes u)}_{\text{divergence-form}}$$

$$\text{Burger: } u_t - \Delta u = - u \cdot \nabla u$$

$$u = U + B(u, u), \quad u_0 \in L^\infty_x, \quad U \in L^\infty_x$$

$$\mathcal{X}_T = L^\infty_{t,x} \cap \{ \nabla u \in L^\infty_{t,x} \}$$

$$\|u\|_{\mathcal{X}_T} = \|u\|_{L^\infty} + \|\nabla u\|_{L^\infty}$$

→ no energy inequality.

$$u_{tt} + u_{xixi} - \Delta u = 0 \Rightarrow u \text{ satisfies the maximum principle.}$$

$$\|u\|_{L_{t,x}} \leq \|u_0\|_{L_x^2} \rightarrow \text{have global existence}$$

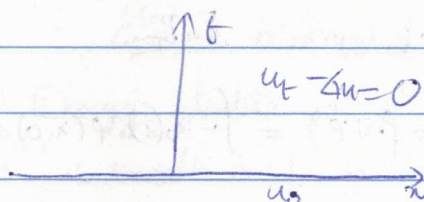
$\omega_0 \in L^{\infty}$

Note: for $n=2$, NSE has another conserved quantity $w = \text{curl } u$
 $w = \text{curl } u$ (vorticity).

$$w_t + (u \cdot \nabla) w = \Delta w$$

03/24/2014

Weak solutions



model: $u_t + g(x,t)u_{xx} - \Delta u = \text{div} f$. (1), $\text{div} a = 0$

Connective with the perturbation theory:

$$u_t^{(n+1)} + \nabla p^{(n+1)} - \Delta u^{(n+1)} = -\frac{\partial}{\partial x_j} u_j^{(n)} u^{(n)}, \quad u_{n+1}^{(n+1)}(x, 0) = u_0(x)$$

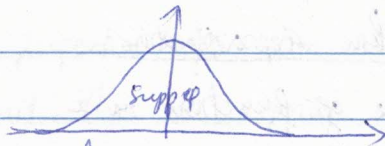
→ can invert to find $u^{(n+1)}$, but ^{only} doesn't satisfy energy estimate

Instead: $y^{(n+1)} + \nabla p^{(n+1)} - \Delta u^{(n+1)} = -\frac{\partial}{\partial y} y^{(n)} u^{(n+1)}$.

This is of this form $u^{(n+1)} + \nabla p^{(n+1)} - \Delta u^{(n+1)} + y_i \frac{\partial u^{(n+1)}}{\partial y_i} = 0$

$$u^{(n+1)} = U + B(u^{(n)}, u^{(n+1)})$$

Test function $\varphi(x, t)$



Several definition of weak solutions

a) "very weak solution" of (1):

$$\iint [u(\varphi_t - g \varphi_j - \Delta \varphi) + f \cdot \nabla \varphi] dx dt = \int -u_0 \varphi(x, 0) dx$$

Need: $u \in L^1_{t,x \text{ loc}}$, $u, g \in L^1_{t,x \text{ loc}}$ to make sense of the above equation.

(b) Multiplying both sides of (1) by u and taking the integration in x :

$$\partial_t \int \frac{|u|^2}{2} dx + \int |\nabla u|^2 dx = \int -f \cdot \nabla u dx$$

$$\|u\|_T^2 \leq \|f\|_{L^2_{t,x}(\mathbb{R}^n \times [0,T])}$$

"weak solution" to (1):

$$u \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1, \text{ whose norm is } \|u\|_T.$$

$$\iint (u \cdot \varphi_t - g \cdot u \cdot \varphi_j - \nabla u \cdot \nabla \varphi + f \cdot \nabla \varphi) = \int -u_0(x) \varphi(x, 0) dx$$

(c) "weak solution" 2

$$u \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1, \quad u_t \in L_t^2 \dot{H}_x^{-1}$$

Comment: for def. (a), the existence is easy to prove, but the uniqueness is difficult to show. For def. (c), the situation is reversed because we imposed many conditions.

A fundamental problem to NSEs is to identify a class in which we can show both the existence and uniqueness. The uniqueness give the equation the predictive power. This is a more fundamental problem than the singularity problem, which is a Millennium problem.

Galerkin approximation

V_m subspace of functions in x .

$\phi_1, \dots, \phi_m$ basis

$$u^{(m)}(x, t) = \sum c_k^{(m)}(t) \phi_k(x)$$

$$u^{(m)}: (0, T) \rightarrow V_m$$

"Project" the equation on V_m :

$$\frac{\partial}{\partial t} \int u^{(m)}(x, t) \varphi(x) + \int -g_j u^{(m)} \varphi_{ij} + \int \nabla u^{(m)} \cdot \nabla \varphi = \int -f_j \varphi_{ij}$$

$$\forall \varphi \in V_m \quad (2)$$

Equivalently, it must be true for $\varphi = \phi_1, \dots, \phi_m$.

$\rightarrow$ get ODE for $c_k^{(m)}(t)$,

$$c^{(m)} = \begin{pmatrix} c_1^{(m)} \\ \vdots \\ c_m^{(m)} \end{pmatrix}$$

$\rightarrow$ get local existence

$$\dot{c}_k^{(m)} = M_{k1}^{(m)}(t) c_1^{(m)}(t) + N_k^{(m)}(t)$$

1) Energy identity for $u^{(m)}$:

$u^{(m)}$ is smooth in t , so (2) $\Rightarrow$

$$\int u_t^{(m)}(x, t) \varphi(x) + \int -g_j u^{(m)} \varphi_{ij} + \int \nabla u^{(m)} \cdot \nabla \varphi = \int -f_j \varphi_{ij} \quad \forall \varphi \in V_m$$

Replace φ by $u^{(m)}$: $\|u^{(m)}\| \leq \|u_0\|_{L^2} + \|f\|_{L^2(\mathbb{R}^n \times (0, T))}$

Initial condition: $\int u_0^{(m)} \phi_k = \int u_0 \phi_k$

independent of m .

$(u^{(m)})$ bounded in $L^2_{t,x}(\mathbb{R}^n \times (0, T))$.

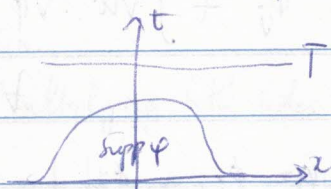
$$u^{(m)} \rightharpoonup u$$

Then u is a weak solution (not difficult to show). The energy estimate is still true because the norm is lower semicontinuous. But we lose all other properties/information about u because of the passing through the limit. This is a price we pay compared to the mild solution approach (iteration).

03/26/2014

$$\iint -u^{(m)} \varphi_t^{(m)} + g_{ij} u^{(m)} \varphi_{ij}^{(m)} - u^{(m)} \Delta \varphi + f \varphi = \int u_0 \varphi^{(m)}(x, 0) dx \quad (*)$$

$$\varphi^{(m)}: [0, T] \rightarrow V_m, \quad u^{(m)} \xrightarrow{\text{weak}} u \text{ in } L^2_{loc}.$$



$$? \text{ Find } \varphi^{(m)}: [0, T] \rightarrow V_m, \quad \varphi^{(m)}(x, t)$$

$$\left. \begin{aligned} \varphi_t^{(m)} &\rightarrow \varphi_t \\ \nabla \varphi^{(m)} &\rightarrow \nabla \varphi \\ \Delta \varphi^{(m)} &\rightarrow \Delta \varphi \end{aligned} \right\}$$

φ is a priori given, $(\varphi^{(m)})$ is to be chosen appropriately (for example the Fourier basis)

Lemma: $f_n, g_n \in H$, $f_n \rightarrow f$, $g_n \rightarrow g$, then $(f_n, g_n) \rightarrow (f, g)$.

Different form:

$$\iint -u^{(m)} \varphi_t^{(m)} + g_{ij} u^{(m)} \varphi_{ij}^{(m)} + \int \nabla u^{(m)} \nabla \varphi^{(m)}$$

In this form, we only need $\left\{ \begin{aligned} \varphi_t^{(m)} &\rightarrow \varphi_t \\ \nabla \varphi^{(m)} &\rightarrow \nabla \varphi \end{aligned} \right.$

If these strong convergences are satisfied then take $m \rightarrow \infty$ in (*), get weak equation for u .

$\Rightarrow$ the existence of weak solution is not difficult to show.

Uniqueness in the energy class

$\|u\| \leq \infty$. This is not the most powerful method for linear equations.

The most powerful one for linear equations is the duality method, which was discussed a while ago. But it is fine for NSE.

$$u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1$$

$$\iint [-u_t \varphi + \nabla u \cdot \nabla \varphi + g_{ij} u_{ij} \varphi + f \cdot \nabla \varphi] dx dt = \int u_0 \varphi(x, 0) dx$$

Uniqueness for smooth solution: choose $\varphi = u \psi_R$, $R \rightarrow \infty$

cut-off function

$$\begin{array}{c} 0 \\ \begin{array}{|c|} \hline \psi_R = 1 \\ \hline \end{array} \\ 0 \quad R \end{array}$$

$$? \int_{\mathbb{R}^n} u u_t dt = \frac{\partial}{\partial t} \int_{\mathbb{R}^n} \frac{u^2}{2}$$

Recall class "(C)" $u_t \in L_t^2 H_x^{-1}$

$$\|w\|_{H^{-1}} = \sup_{\|w\|_{H^1} \leq 1} \int \nabla \cdot w$$

$$\|w\|_{H^{-1}} = c \int |\xi|^{-2} |\hat{w}(\xi)|^2 dt$$

$$u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1$$

$$u_t \in L_t^2 H_x^{-1}$$

Assume $u \in L_t^2 H_x^1$, $u_t \in L_t^2 H_x^{-1}$,

$$v \in L_t^2 H_x^1, v_t \in L_t^2 H_x^{-1}$$

$\iint u \cdot v_t dx dt$ is a well-defined expression

If $\int u(x,t) v(x,t) dx$ is also a continuous function of t ,

$$\frac{\partial}{\partial t} \int u(x,t) v(x,t) dx = \int (u_t(x,t) v(x,t) + u(x,t) v_t(x,t)) dx$$

L^1 -function of t .

$$\text{Apply: } \int_{\mathbb{R}^n} u u_t dt = \frac{\partial}{\partial t} \int_{\mathbb{R}^n} \frac{u^2}{2}$$

When $u \in L_t^2 H_x^1 \cap L_t^\infty L_x^2$, $u_t \in L_t^2 H_x^{-1}$

$$u \in C_t L_x^2$$

$$\int \frac{|u(x, t_2)|^2}{2} dx - \int \frac{|u(x, t_1)|^2}{2} dx = \int_{t_1}^{t_2} \underbrace{\int_{\mathbb{R}^n} u u_t dt}_{\in L_t^1}$$

When $t \rightarrow t_1$, $\|u(\cdot, t)\|_{L_x^2}^2 \rightarrow \|u(\cdot, t_1)\|_{L_x^2}^2$. Thus $u(\cdot, t) \rightarrow u(\cdot, t_1)$

$$\begin{aligned} \int (u(x, t) - u(x, t_1)) \psi(x) dx &\rightarrow 0 \quad \forall \psi \\ &= \int_{t_1}^t u_t(x, s) \psi(x) ds \end{aligned}$$

Note: the assumption $u_t \in L_t^2 H_x^{-1}$ gives some regularity in t -direction.
If we only have $u \in L_t^2 H_x^1 \cap L_t^\infty L_x^2$, we can choose $u(x, t) = g(t) \psi(x)$

$$u \in L_t^2 H_x^1 \quad u(x, t) = \sum_n \hat{u}_n(t) e^{inx}$$

$$\sum_n |\hat{u}_n(t)|^2 < \infty \quad L_t^\infty L_x^2$$

$$\sum_n \int |\hat{u}_n(t)|^2 dt < \infty \quad L_t^2 H_x^{-1}$$

$\rightarrow$ force each Fourier component to be continuous.

Note: if $u_t - \Delta u = f \in L^2$ then $u_t, \nabla^2 u \in L^2$

but for $u_t - \Delta u = \text{div} f$, $u_t = \text{div}(\nabla u f)$. If $f \in L^2$, it is natural that $\nabla u \in L^2$, but not $u_t \in L^2$. Instead, $u_t \in \mathcal{D}' = \text{div} F$, $F \in L^2$. Note: $\{\text{div} F \mid F \in L^2\} = H^{-1}$

$\Rightarrow$ the condition $u_t \in L_t^2 H_x^{-1}$ is natural for $u_t - \Delta u = \text{div} f$.

Conceptual:

$$\partial_t \quad \partial_x^2 \quad \text{"}\partial_t^{1/2}\text{" } u \in L^2$$

$$\int u u_t \text{ "}" \int u \partial_t u \text{ "}" \int \underbrace{(\partial_t^{1/2} u)}_{\in L^2}, \underbrace{(\partial_t^{1/2} u)}_{\in L^2}$$

Question: With $u_0, f \in L^2(\mathbb{R}^3)$, and u obtained in the weak formulation

can we show that $u_t \in L_t^2 H_x^1$, which we assumed last time?

3/28/2014

~~Answer~~ Answer to the question: YES.

$$u_t - \Delta u = \text{div } f, \quad f \in L^2$$

Fourier transform for space-time: $(i\tau + |\xi|^2) \hat{u} = i\xi_k \hat{f}_k$

$$\Rightarrow \hat{u} = \frac{i\xi_k \hat{f}_k}{i\tau + |\xi|^2} = \underbrace{\frac{i\xi}{i\tau + |\xi|^2}}_{\text{not bounded in } 0} \hat{f}$$

$\Rightarrow$ even if $\hat{f} \in L^2$, $\hat{u} \notin L^2$.

Now why $u_t \in L_t^2 H_x^1$?

by Fourier transform: $\partial_t \leftrightarrow i\tau$
 $H^1 \leftrightarrow \frac{1}{|\xi|}$

$$\frac{i\xi}{i\tau + |\xi|^2} \frac{i\tau}{|\xi|} = -\frac{\xi}{|\xi|} \frac{\tau}{i\tau + |\xi|^2} \quad \text{is bounded}$$

$$\|\partial_t^{1/2} u\| \longleftrightarrow \frac{i|\tau|^{1/2} \xi}{i\tau + |\xi|^2} \quad \text{bounded} \Rightarrow \partial_t^{1/2} u \in L^2$$

$$\nabla u \longleftrightarrow \frac{i|\xi| \xi}{i\tau + |\xi|^2}$$

$$\partial_x \longleftrightarrow i\xi$$

$$|\partial_x|^{1/2} \longleftrightarrow |\xi|^{1/2}$$

Why $u \in L_t^\infty L_x^2$? explain in Fourier picture:

$$\widehat{u(x,0)} = \widehat{u(x,t)} \widehat{S(t)} = \hat{u} * \hat{S}$$

Definition of Fourier transform: in Hörmander, Physics

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\xi} dx$$

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\xi} d\xi$$

$$\int f \bar{g} = \frac{1}{(2\pi)^n} \int \hat{f} \overline{\hat{g}}$$

Harmonic analysis (Stein, Tao, ...):

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi x} dx$$

Engineering: $f(x) = \int \hat{f}(\xi) e^{i x \xi} d\xi$

$$\widehat{u(x,0)} = c \int_{-\infty}^{\infty} \hat{u}(\xi, z) dz$$

$$\hat{v}(\xi) = \int_{-\infty}^{\infty} \frac{i\xi}{z^2 + |\xi|^2} \hat{f}(\xi, z) dz$$

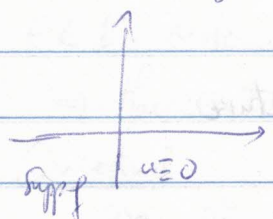
$$\Rightarrow \|\hat{v}(\xi)\|_{L^1} |\hat{v}(\xi)| \leq \left\{ \int_{-\infty}^{\infty} \frac{|\xi|^2}{z^2 + |\xi|^4} dz \right\}^{1/2} \left\{ \int_{-\infty}^{\infty} |\hat{f}(\xi, z)|^2 dz \right\}^{1/2}$$

$$\Rightarrow \|\hat{v}(\xi)\|^2 \leq \underbrace{\int_{-\infty}^{\infty} \frac{|\xi|^2}{z^2 + |\xi|^4} dz}_{\text{bounded (uniformly in } \xi)} \int_{-\infty}^{\infty} |\hat{f}(\xi, z)|^2 dz \Rightarrow \|u(\cdot, 0)\|_{L^2} \leq \text{const}$$

(Then do the same thing for other time slices.)

What is the easiest way to see $u \in C_t L_x^2$?

The best way is from the energy estimates.



$$\|u(\cdot, t_1)\|^2 \leq c \int_{t \leq t_1} \int_{\mathbb{R}^n} |f|^2 dz dt$$

$$\|u(\cdot, t_1+h) - u(\cdot, t_1)\|_{L^2}^2 \leq c \int_{t \leq t_1} \int_{\mathbb{R}^n} |f(x, t_1+h) - f(x, t)|^2 dz dt$$

solves a heat equation
with the righthand side $f(\cdot, t_1+h) - f(\cdot, t_1)$

$$\left\{ \begin{array}{l} h \rightarrow 0 \\ \Downarrow \\ \|f\|_{C^2}^2 \end{array} \right.$$

Return to Galerkin's method:

$$\int u_t^{(m)} v = \int [-a_j u^{(m)} - u_{ij}^{(m)} - f_j] v_{,j}$$

$$\|u^{(m)}\| \leq C.$$

$$\|u_t^{(m)}\|_{L^2 H^{-1}} \leq \tilde{C} (?)$$

$u(x,t)$... any function. (locally integrable)

$$\frac{\partial}{\partial t} u = \operatorname{div} F \quad (\text{in sense of distribution})$$

$F \in L^2_{t,x}$

$$\Rightarrow u_t \in L^2_{t,x} H^{-1}_x$$

$$\int -u \varphi_t = \int -F_j \frac{\partial \varphi}{\partial x_j}$$

We need $-a_j u^{(m)} - u_{ij}^{(m)} - f_j \in L^2_{t,x}$. We have

$$\|u^{(m)}\| \leq C, \quad \|u^{(m)}\|_{L^{\frac{2^*}{n}}_{t,x}} \leq \tilde{C}$$

(embedding)

where

$$\frac{1}{\frac{2^*}{n}} = \frac{1}{2} - \frac{1}{n+2}.$$

If we want $a_j u^{(m)} \in L^2$, while $u^{(m)} \in L^{\bar{p}}$ ($\bar{p} = \frac{2^*}{n}$), then we require $a_j \in L^q$ where $\frac{1}{q} + \frac{1}{\bar{p}} = \frac{1}{2}$.

$$\Rightarrow \frac{1}{q} + \frac{1}{2} - \frac{1}{n+2} = \frac{1}{2} \Rightarrow q \geq n+2$$

$\Rightarrow$ for the purpose of getting energy estimate (and then the unique^{ness} of weak solutions), we need $a_j \in L^{n+2}$. This is the critical space.

If we need the existence of weak solution only; we only require $a_j u^{(m)} \in L^1$. Since $u^{(m)} \in L^{\bar{p}}_{t,x}$, we require $a_j \in L^{\bar{p}'}_{t,x}$ where

$$\bar{p} \geq \bar{p}', \quad \frac{1}{\bar{p}'} + \frac{1}{\bar{p}} = 1 \Rightarrow \bar{p}' = \frac{2(n+2)}{n+4}.$$

Uniqueness of weak solution: $q \geq n+2$

(existence via perturbation)

Existence of weak solution: $r \geq \bar{p}' = \frac{2(n+2)}{n+4}$

Note: For bounded domain, we can allow $q > n+2$. But for the whole space, we do need $q = n+2$.

03/31/2019

regularity?

$a_j u_j$

$$u_t + a_j u_j - \Delta u = \text{div } g$$

$$1) a_{j,j} = \text{div } a = 0$$

$$2) a_j u \in L^1_{\text{loc}}, \quad \|u\| < \infty$$

$$\text{Recall that we wrote } \int a_j u_j \varphi \, dx = \int -a_j u \varphi_{,j} \, dx$$

$$= (a_j u)_{,j} - \underbrace{a_{j,j}}_{=0} u$$

$$u \in L^p_{\text{loc}}, \quad \frac{1}{p} = \frac{1}{2} - \frac{1}{n+2} \Rightarrow \text{require } a_j \in L^{p'}_{\text{loc}}$$

$$3) u_t \in L^2_{\text{loc}} H^{1,1}_{\text{loc}} \Rightarrow \text{require } a_j u \in L^2_{\text{loc}} \text{ loc} \\ \Rightarrow \text{require } a_j \in L^{n+2}_{\text{loc}}$$

Aside: $u = (u_1, u_2)$, $\det Du := u_{1,1} u_{2,2} - u_{1,2} u_{2,1}$.

or we can define $\det Du := \frac{\partial}{\partial x_1} (u_1 u_{2,2}) - \frac{\partial}{\partial x_2} (u_1 u_{2,1})$

$$-\Delta u + a_j u_j = 0 \\ \downarrow \\ \in L^n_{\text{loc}}$$

$$-\Delta u + a_j u_j = u_t \\ \downarrow \\ \in L^{n+2}_{\text{loc}}$$

Return to Galerkin: $\int u_t^{(m)} v^{(m)} = \int [a_j u^{(m)} - u_{,j}^{(m)} - f_{1,j}] v_{,j}^{(m)}, \quad v^{(m)} \in V$

Limit: $\iint -u \varphi = \iint \underbrace{[a_j u - \dots]}_{\in F = (f_j) \in L^2_{\text{loc}}} \varphi_{,j}$

$$u_t = -\operatorname{div} F \in L^2_{loc}$$

$$u_t = L^2_t H^1_x$$

How to get the estimate $\|u_t^{(m)}\|_{L^2_t H^{1-1}_x} \leq C < \infty$?

$$\int u_t^{(m)} \varphi(x) dx = \int u_t^{(m)} v^{(m)} dx + \int u_t^{(m)} w^{(m)} dx$$

$$(\text{where } \varphi = v^{(m)} + w^{(m)})$$

$\hookrightarrow L^2$ -projection of φ onto $V^{(m)}$

$$= \underbrace{\int [\dots] v_{ij}^{(m)}}_{\text{use [1] estimate}} + \underbrace{\int [\dots] w_{ij}^{(m)}}_{\text{use } L^2\text{-estimate} = 0}$$

$$\|\nabla v^{(m)}\| \leq C \|\nabla \varphi\|, \quad \|\nabla w^{(m)}\| \leq C \|\nabla \varphi\|.$$

Need: $P_m: D(\mathbb{R}^n) \rightarrow V^{(m)}$ (L^2 -projection)

$$\|P_m \varphi\|_{H^1_x} \leq C \|\varphi\|_{H^1_x}$$

$\uparrow$
independent of m .

$$\begin{aligned} \int [a_j u^{(m)} - a_{ij}^{(m)} - f_{ij}] v_j^{(m)} &\leq \|[\dots]\|_{L^2_x} \|\nabla v_j^{(m)}\|_{L^2_x} \\ &\leq \|\nabla v^{(m)}\|_{L^2_x} \leq C \|\nabla \varphi\|_{L^2_x} \end{aligned}$$

What will we do next?

$$1) \quad u_t + a_j u_j - \Delta u = \operatorname{div} f, \quad \operatorname{div} a = 0$$

global existence + uniqueness of weak solution

$$a_j \in L^{n+2}_{t,x}$$

2) Regularity:

$u^{(1)} = U$ heat equation of u_0

$$u_t^{(m+1)} + u^{(m)} \cdot \nabla u^{(m+1)} - \Delta u^{(m+1)} + \operatorname{div} p^{(m+1)} = 0$$

$$\int u^{(m)} \cdot \nabla u^{(m+1)} u^{(m+1)} = 0$$

$$\|u^n\| \leq C < \infty.$$

$$u^{(m)} \rightarrow u$$

04/02/2014 Discuss the equation last time: regularity of $a_j(x)u_j$

$$-\Delta u + a_j(x)u_j + b(x)u = 0$$



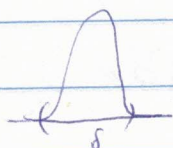
$$x = \delta y$$

$$u_\delta(y) = u(\delta y)$$

$$a_{\delta j}(y) = \delta a_j(\delta y)$$

$$b_\delta(y) = \delta^2 b(\delta y)$$

$$\text{then } -\Delta u_\delta + a_{\delta j} u_{\delta j} + b_\delta u_\delta = 0.$$



$$u \sim 1$$

$$\nabla u \sim \delta^{-1}$$

$$\nabla^2 u \sim \delta^{-2}$$

$$a_j \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

$$\text{"Borderline": } a_j \sim \frac{1}{|x|}$$

$$\|a_j\|_{L^p(B_1)} = \|a_j\|_{L^p(B_\delta)} \Rightarrow p = n.$$

$$\text{"Borderline"} \quad b(x) \sim \frac{1}{|x|^2}$$

$$L^p: b \in L^{n/2}$$

These are also the borderline for large scale ($\delta \gg 1$).

$$-\Delta u_\delta + a_{\delta j} u_{\delta j} = 0$$

the solution will be continuous / Hölder continuous if $\|a_j\|$ is small whenever δ is small.

Weak solutions of Navier-Stokes

(1) Galerkin approximation:

$$V^m \subset \{u: \mathbb{R}^n \rightarrow \mathbb{R}^n, u \text{ smooth, rapidly decaying at } \infty, \operatorname{div} u = 0\}$$

$$\dim V^m < \infty.$$

Seek $u^m: [0, T] \rightarrow V^m$, absolutely continuous

$$\int (u_t^m \cdot v + u^m \nabla u^m \cdot \nabla v + \nabla u^m \cdot \nabla v) = 0 \quad \forall v \in V^m \quad (*)$$

Condition on v : $v \in V^m, \nabla v \in L^2$.

$$\text{Fix } m: u^m = \sum c_k^m(t) \underbrace{\phi_k^m}_{\text{basis}}$$

ODE for $\rightarrow$ get local-in-time existence.

Take $u(x) = u^m(x, t) \Rightarrow$ energy identity

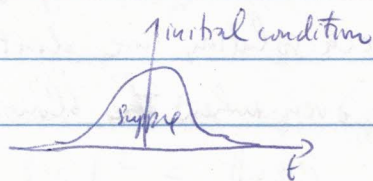
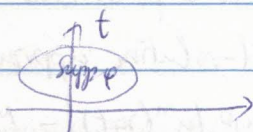
$\|u^m\| \leq C$ (independent of m , on $(0, T)$ for any $T > 0$).

$$u^m \rightharpoonup u, \nabla u^m \rightharpoonup \nabla u \quad (\text{for a subsequence})$$

Question: show that u is a weak solution of the original equation.

$$\iint (-u \varphi_t - u u_x \varphi_x + \nabla u \cdot \nabla \varphi) dx dt \quad \forall \varphi, \operatorname{div} \varphi = 0$$

$$= \int_0^\infty \varphi(x) dx$$



$$\iint [-u^m \cdot v_t - \underbrace{(u^m \otimes u^m)}_{??} \nabla v + \nabla u^m \cdot \nabla v] dx dt = \int \text{initial data} \quad (*)$$

$\downarrow m \rightarrow \infty$ $\downarrow m \rightarrow \infty$
 no problem no problem

$$f_m(x) = \sin(mx) \psi(x) \rightarrow 0$$

$$f_m f_m = f_m^2 \rightarrow \frac{1}{2} \psi(x)^2$$

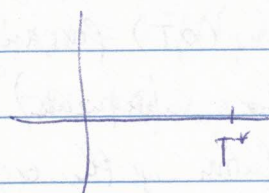
Problem: how to control the oscillation in t .

$\|\nabla u^m\|_{L^2}^2 \rightarrow$ "control ~~partial~~ spatial oscillation" only.
 To control the oscillation int, we have to use the equation.
 Sometimes, it is better to not impose $\operatorname{div} u^m = 0$. Instead, allow $\operatorname{div} u^m = g^m$ which converges to 0.

Other approximations

$$\begin{cases} u_t^{n+1} + u^n \cdot \nabla u^{n+1} + \nabla p^{n+1} - \Delta u^{n+1} = 0 \\ u^{(m+1)}(x, 0) = u_0 \\ \operatorname{div} u = 0 \end{cases}$$

This is a PDE, looking more advantageous than (*).

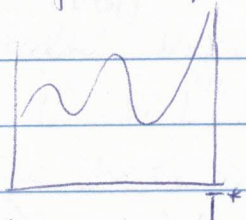


$$\|\nabla u^n\| \leq C < \infty$$

In any method of approximation, we always run into the problem of passing the limit of the nonlinear term $u_m \otimes u_m$.

04/04/2014 Question: Does the weak formulation reflect the blowing-up phenomenon as in mild-solution approach?

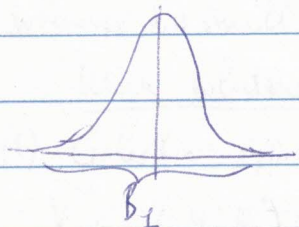
Yes! for example $\sup |u^m(x, t)| = M_m(t)$ may go to ∞ as $m \rightarrow \infty$.



For weak solution, we don't encounter this issue even when the blowing up really happens.

Leray's approximation: $(u \cdot \nabla) u \rightarrow (u^n \cdot \nabla) u^{n+1}$

ϕ : mollifier (in space) $\int \phi = 1$, $\phi_\varepsilon = \varepsilon^{-n} \phi\left(\frac{x}{\varepsilon}\right)$



$$u \rightarrow \phi_\varepsilon * u = (u)_\varepsilon$$

Leray approximation:

$$u_t + (u)_\varepsilon \nabla u + \nabla p - \Delta u = 0 \quad (*)_\varepsilon$$

1) Mild solution; quite the same (or even easier) as in $u_t + (u_0)u + \nabla p - \Delta u = 0$, which was done before.

2) If $u_0 \in L_x^\infty \cap L_x^\infty$ (or $u_0 \in L_x^2 \cap L_x^\infty$) then we get the energy estimate $\|u\| \leq C < \infty$ (for any time interval $[0, T]$).

$$\Rightarrow \|u\|_{L_t^\infty L_x^2}^2 \leq C$$

$$\Rightarrow \|(u)_\varepsilon\|_{L_x^\infty} \leq C(\varepsilon)$$

$$\| (u)_\varepsilon \|_{L_x^2} \leq C$$

$(u)_\varepsilon = a(x, t)$ is a smooth function.

$$|\nabla_x^m a| \leq C_m(\varepsilon),$$

$$\| \nabla_x^m a \|_{L_x^p} \leq C_m(\varepsilon, p), \quad p \geq 2.$$

Equation $(*)_\varepsilon$ becomes $u_t + a \nabla u - \Delta u + \nabla p = 0$.

Note that if u is a solution of the above equation then cu is also a solution $\Rightarrow$ can control the initial data to be as small as possible we want.

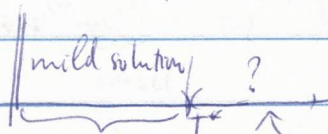
 $\Rightarrow$ get global existence for $(*)_\varepsilon$.

$(*)_\varepsilon$ has solution u_ε (global solution)

$$\|u_\varepsilon\|_{L_t^2 L_x^2}^2 \leq \|u_0\|_{L_x^2}^2$$

$$f \equiv 0$$

$$u_\varepsilon \rightarrow u \text{ (subsequence)}$$

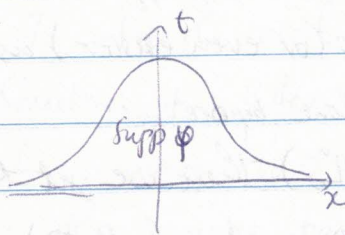


$$u_\varepsilon \rightarrow u \text{ as } \varepsilon \rightarrow 0$$

weak solution exists here also, but what is it?

Leray's method relates to "LES" (local estimate solution) existence?

Claim: u is a weak solution of the NS for any time interval

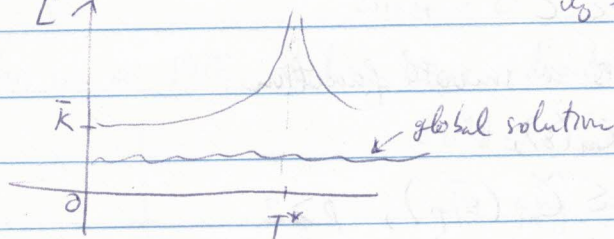


$$\iint -u \varphi_t - u_j \varphi_{tj} + \nabla u \cdot \nabla \varphi = \int -u_0 \varphi(x, 0) dx$$

u^m ... sequence of smooth solution to NSE,
 $\|u^m\| \leq C, u^m \rightharpoonup u$

u ... weak solution

" L^2 "



$$u_0 \mapsto \kappa u_0$$

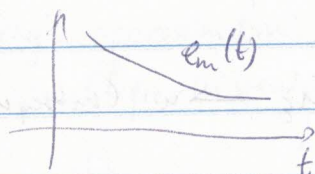
For $0 < \kappa < \bar{\kappa}$, the global solution exists.

$$\begin{aligned} \iint -u^m \varphi_t - u^m_j \varphi_{tj} + \nabla u^m \cdot \nabla \varphi &= \int -u_0^m \varphi(x, 0) dx \\ \downarrow \text{m.s.s} & \quad \downarrow ?? \quad \downarrow \\ \iint -u \varphi_t & \quad \iint \nabla u \cdot \nabla \varphi \end{aligned}$$

Leray (1934): denote $e_m(t) = \int |u_m(x, t)|^2 dx$

$$D_m(t) = \int |\nabla u_m(x, t)|^2 dx$$

$$\dot{e}_m = -D_m \leq 0$$



$\exists$ subsequence $e_{m_k}(t) \rightarrow e_*(t)$ pointwise
 not necessarily

continuous in t.

$$D_*(t) = \liminf D_m(t)$$

$$\int_{t_1}^{t_2} \liminf_{D_*} D_m \leq \liminf \int D_m$$

$\Rightarrow D_*$ is integrable ($\Rightarrow$ finite almost everywhere)

Put $e(t) = \int |u_m(t)|^2 dx$

Main point: $\varphi(x)$... divergence-free test function.

$$\int u_m(x,t) \varphi = \frac{\partial}{\partial t} \int u_m(x,t) \varphi(x) dx$$

= function in L^p

$\Rightarrow \int u_m(x,t) \varphi(x) dx$ is abs. continuous in t (independent of m)

$u_m \rightarrow u$ and $\underbrace{\forall t, u_m(\cdot, t) \rightarrow u(\cdot, t)}_{\text{crucial step}}$

This rules out the possibility $\sin(mt) \varphi(x)$ (no oscillation in t anywhere)

$D_*(t) < \infty$, $u_m(\cdot, t) \rightarrow u(\cdot, t)$

Subsequence m' : $\|\nabla u_{m'}\|_2 \leq C < \infty$

$\Rightarrow$ by embedding theorem (precompact), $u_{m'}(\cdot, t) \rightarrow u(\cdot, t) \forall t$

$\Rightarrow e(t) = e_*(t)$

04/08/2014

$u^m: \mathbb{R}^n \times (t_1, t_2) \rightarrow \mathbb{R}^n$

1) $t \mapsto (u^m(t), v(x))$ is uniformly continuous in t , $\forall v \in L_x^2$.

$\rightarrow$ suffices to show for a dense subset of L_x^2 .

$\left(v = \underbrace{v^{(1)} + v^{(2)}}_{\text{small}} \Rightarrow t \mapsto (u^m(t), v) = \underbrace{(u^m(t), v^{(1)})}_{1-1 < \varepsilon} + \underbrace{(u^m(t), v^{(2)})}_{\text{small}} \right)$

$v = \varphi(x)$

$\varphi = \underbrace{\varphi^{(1)}}_{\text{div}=0} + \nabla \phi$ Helmholtz decomposition

$(u^{(m)}(t), \varphi) = (u^{(m)}(t), \varphi^{(1)}) \Rightarrow$ we can assume that:

φ smooth, $\operatorname{div} \varphi = 0$, "some decay" at ∞

$$t \mapsto (u^{(m)}(t), \varphi) : \underbrace{\frac{d}{dt} \int u^{(m)} \varphi}_{f_m^{(1)}(t)} = \underbrace{\int (-\nabla u^{(m)} \cdot \nabla \varphi + u^{(m)} \varphi_{,ij})}_{= g_m^{(1)}(t)} \leq \underbrace{C \|\nabla u^{(m)}\|_{L^2}}_{= g_m^{(1)}(t)} \leq \underbrace{C \|u^{(m)}\|_{L^2}^2}_{= g_m^{(2)}(t)}$$

We have $\|g_m^{(1)}\|_2 \leq C$

$$|f_m^{(1)}(t)| \leq \underbrace{C \varphi}_{L^2\text{-bounded}} \underbrace{g_m^{(1)}}_{\text{bounded}} + \underbrace{\tilde{C} \varphi}_{\text{bounded}} \underbrace{g_m^{(2)}}_{\text{bounded}}$$

modules of continuity independent of m .

$$u^{(m)}(x, 0) = u_0(x)$$

$u(\cdot, t) \rightarrow u_0(x)$ as $t \rightarrow 0$ (because we controlled everything in time)

Assume the right hand side $(f) \equiv 0$, then

$$\|u^{(m)}(\cdot, t)\|_{L^2} \leq \|u_0\|_{L^2}$$

$$\Rightarrow \|u(\cdot, t)\|_{L^2} \leq \|u_0\|_{L^2}$$

Recall: if $\xi_n \rightharpoonup \xi$ in a Hilbert space and $\|\xi_n\| \leq \|\xi\|$ then $\xi_n \rightarrow \xi$.
Therefore, $u_m(t) \xrightarrow{L^2} u_0$ as $t \rightarrow 0$.

Consider
$$e_m(t) = \int |u_m(x, t)|^2 dx$$

Helly's selection theorem (wiki)

$$\leftarrow e_*(t) = \lim e_m(t) \quad (\text{subsequence})$$

$$e(t) = \int |u(x, t)|^2 dx \quad \left(\begin{array}{l} u_m \rightharpoonup u \\ \text{in } L^2_{b,x} \end{array} \right)$$

$$p_m(t) = \int 2 |\nabla u_m(x, t)|^2 dx$$

$$p_*(t) = \liminf \int |\nabla u_m(x, t)|^2 dx$$

$D_*(t) < \infty$ for almost every t .

$$\frac{u_m(\cdot, t) \rightarrow u(\cdot, t) \quad \forall t}{D_*(t) < \infty}$$

$$\int |\nabla \underbrace{u_{m_i}(\cdot, t)}_{\text{subsequence}}|^2 dx \leq C < \infty$$

For bounded domain: $u_m(\cdot, t) \rightarrow u(\cdot, t)$ is strong (up to subsequence).
 $u_m(\cdot, t) \xrightarrow{L^2} u(\cdot, t)$ in L^2

$$\underline{e_*(t) = e(t)}$$

$e_*(t) = e(t)$ for a.e. t

$$\underbrace{\int_{t_1}^{t_2} e_m(t) dt \rightarrow \int_{t_1}^{t_2} e_*(t) dt = \int_{t_1}^{t_2} e(t) dt}_{\text{Lebesgue's Dominated Convergence}}$$

For unbounded domain: things can escape to infinity.

Example $f \in L^2 \cap D(\mathbb{R})$

$f_h(x) = f(x-h)$. Then $f_h \rightarrow 0$ but $f_h \not\rightarrow 0$.

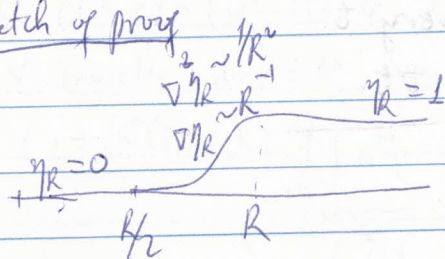


We'll show that this possibility doesn't happen for our solution (by some calculation from the equation).

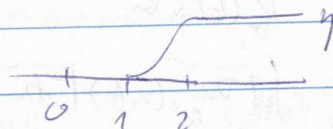
$$\forall \varepsilon > 0, \forall T > 0, \exists R > 0, \int_{\mathbb{R}^n \setminus B_R} |u_m(x, t)|^2 dx < \varepsilon \quad \forall t$$

(energy doesn't escape to infinity)

Sketch of proof



$$\eta_R = \eta\left(\frac{x}{R}\right)$$



$$\partial_t \int \frac{1}{2} |u_m(x,t)|^2 \eta_R(x) dx \leq \int -|\nabla u_m(x,t)|^2 \eta dx + \dots$$

$$\left| \int u_m \cdot \nabla u_m \cdot \nabla \eta_R \right| \leq \|u_m\|_{L_x^2} \|\nabla u_m\|_{L_x^2} \frac{1}{R}$$

$$\int (\nabla u_m \cdot \nabla \eta_R) \left(\frac{|u_m|^2}{2} + p_m \right) = \underbrace{\int (\nabla u_m \cdot \nabla \eta_R) \frac{|u_m|^2}{2}}_{\{1\}} + \underbrace{\int \nabla u_m \cdot \nabla \eta_R p_m}_{\{2\}}$$

The ideas of bounding $\{1\}$ and $\{2\}$ are similar. $\{2\}$ is a bit more difficult than $\{1\}$.

$$|\{1\}| \leq \frac{1}{R} \int |u_m|^3 dx$$

(Recall every estimate comes from the energy estimate. Any other statement than that is unknown!)

$$\|u\|_{L_x^3} \leq \|u\|_{L_x^2}^{1/2} \|u\|_{L_x^6}^{1/2} \quad (n=3)$$

$$\leq \|u\|_{L_x^2}^{1/2} \|\nabla u\|_{L_x^2}^{1/2} \quad (\text{Sobolev's imbedding})$$

Recall $H_x \subset L_x^2$

04/05/2014

$$\begin{cases} u_t + (u)_x \nabla u - \Delta u + \nabla p = 0 \\ u(x,0) = u_0 \in L_x^2 \quad (\text{instead of } L_x^2 \cap H_x^1) \end{cases}$$

Does it have a solution? Prof. Sverak guessed Yes!

Terry Tao in a paper 2 years ago had a paper about this. Look on Arxiv.

Monday, Wednesday: no class

April 14, 16

u^m

$$e_m(t) = \int \frac{1}{2} |u^m(x,t)|^2 dx$$

$$D_m(t) = \int |\nabla u^m(x,t)|^2 dx$$

$$e_m(t_1) - e_m(t_2) = \int_{t_1}^{t_2} D_m(t) dt \quad (1)$$

$u^m \rightarrow u, u^m \xrightarrow{L^2_{loc}} u$ What can be said about u ?

$$e_m(t) \rightarrow e(t), \quad e(t) = \int \frac{1}{2} |u(x,t)|^2 dx$$

$$D_m(t) = \int |\nabla u^m(x,t)|^2 dx \rightarrow \liminf D_m(t)$$

$e_m(t) \rightarrow e(t) = e(t)$ almost everywhere ($\forall t, D_m(t) < \infty$)

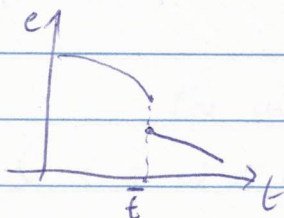
$$(1) \Rightarrow e_m(t_1) \geq e_m(t_2) + \int_{t_1}^{t_2} D_m(t) dt$$

$$e(t_1) \xrightarrow{\text{red}} e(t_1) \quad e(t_2) \geq e(t_2)$$

Question: $e(t_2) + \int_{t_1}^{t_2} D(t) dt \leq e(t_1)$

1) Ok. $\forall$ for $\forall t_2, t_1 = 0$

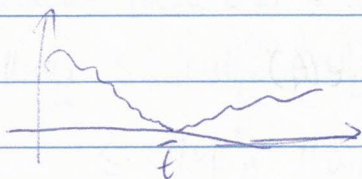
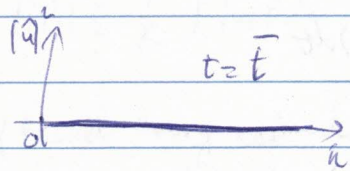
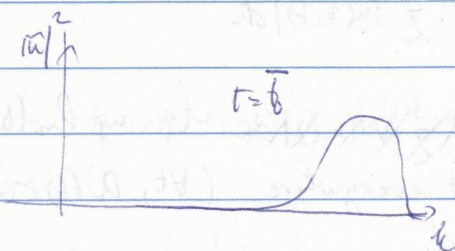
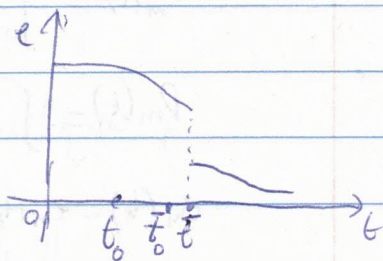
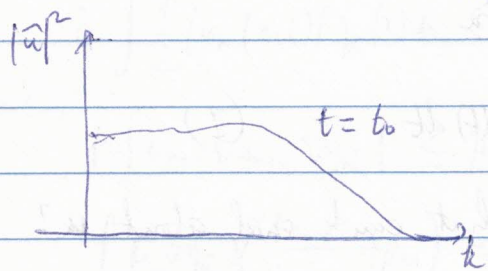
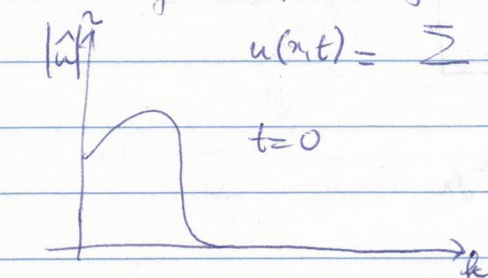
2) Ok $\forall t_2$ with $D_m(t_1) < \infty$,
(i.e. t_1)



The possibility that $e(T) = 0$ has not been ruled out by current theory.

Scenario for this possibility to happen:

$$u(x,t) = \sum \hat{u}(k,t) e^{ikx} \quad (\text{periodic solution, or a wave})$$



Energy Frequency disappear to infinity then come back.

This scenario is not likely to happen, but

is an open problem!

If this happens, then we'll lose the uniqueness of NSE.

Regularity properties

Consider $\bar{t}$ with $D_*(\bar{t}) < \infty$.

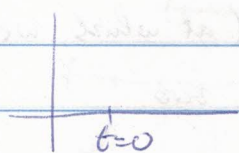
$$\|\nabla u^{m'}(\cdot, \bar{t})\|_{L^2} \leq C$$

$$\|v\|_6 \leq C \|\nabla v\|_2 \Rightarrow \|u^m(\cdot, \bar{t})\|_{L^6} \leq C$$

subcritical

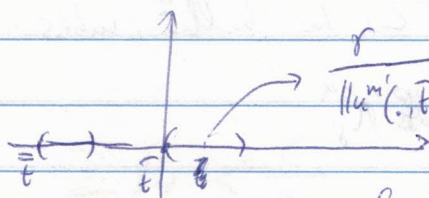
Now regard $\bar{t}$ as 0 and $u_0^m = u(\cdot, \bar{t})$. Then we'll be in a subcritical setting: $\|v_0\|_{L_x^6} < \infty$.

$\Rightarrow$ mild solution exists in $(0, T^*)$



$$T^* \geq \frac{\gamma}{\|v_0\|_{L^6}^4}$$

The quantity $t^{1/4} \|v_0\|_6$ is scaled-invariant. ($\lambda v_0(\lambda x) = v_{0,1}(x)$)



$$\frac{\gamma}{\|u^m(\cdot, \bar{t})\|_{L^6}^4}$$

(γ is fixed)

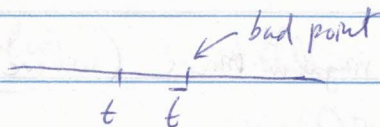
We are fine with passing the limit $m \rightarrow \infty$. There is no pathology in a short interval $(\bar{t}, \bar{t} + \dots)$. In addition, there are almost everywhere such $\bar{t}$.

$$f(t) = D_\lambda(t) = \limsup \int |\nabla u^m(x, t)|^2 dx$$

$$\|u(\cdot, t)\|_{L_x^6}^2 \lesssim f(t)$$

$f \in L^1$, $f(t) < \infty \Rightarrow u$ is "nice" on $(t, t + \tau)$, $\tau \geq \frac{t}{f(t)^2}$

Another way



$$f(t) \geq \frac{c_1}{\sqrt{t-t}}$$

(quite large, otherwise

the short interval $(t, t + \dots)$ would be able to cover $\bar{t}$)

S is a set $\subset (t_1, t_2)$

$f: (t_1, t_2) \rightarrow \mathbb{R}$ integrable.

$$\forall \bar{t} \in S^*, f(t) \geq \frac{\varepsilon_0}{\sqrt{\bar{t}-t}} \quad \forall t < \bar{t}.$$

Lemma: $\mathcal{H}^{1/2}(S) = 0$.

$\Rightarrow$ the set of bad points of weak solution (at which we'll lose the uniqueness) has Hausdorff $1/2$ measure zero.

04/11/2014

$E \subset \mathbb{R}^n$. Fix $\delta > 0$, α -dimensional measure.

$\{B_i\}$ covering of E , B_i can be balls, cubes, ...

$$\sup_{\substack{\text{covering} \\ \text{diam } B_i \leq \delta}} \sum_i (\text{diam } B_i)^\alpha = \mathcal{H}_\delta^\alpha$$

normalization

$$\mathcal{H}^\alpha = \lim_{\delta \rightarrow 0^+} \mathcal{H}_\delta^\alpha$$

Last time: Weak solution u .

$$f(t) = R(t) = \liminf \|\nabla u_m(t)\|_{L^2}^2$$

$\bar{t}$ = possible singular time.

$$f(t) \geq \frac{\varepsilon_1}{\sqrt{\bar{t}-t}} \quad \text{for } t < \bar{t}$$

Recall: $f(t) \in L^1$.

Let S be the set of singular times (where $D_\kappa(t) = \infty$)

Claim: $\mathcal{H}^{1/2}(S) = 0$.

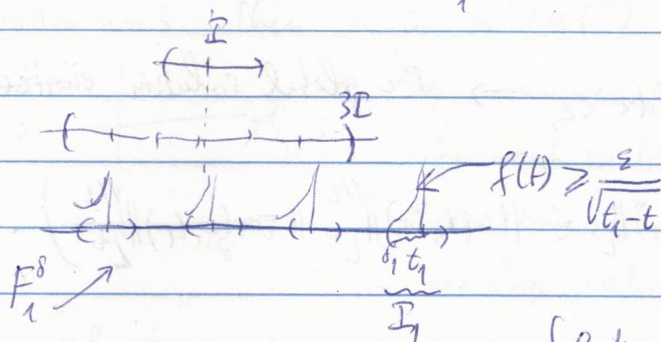
Proof: Let F^δ be balls (in our case, intervals) centered at S , of diam $< 2\delta$.

$$S \subset \bigcup F^\delta$$

Covering lemma: ($n=1$; also true for $\mathbb{R}^n$ with suitable change)

∃ a subsystem F_1^S such that $\forall I, J \in F_1^S, I \cap J = \emptyset$ and

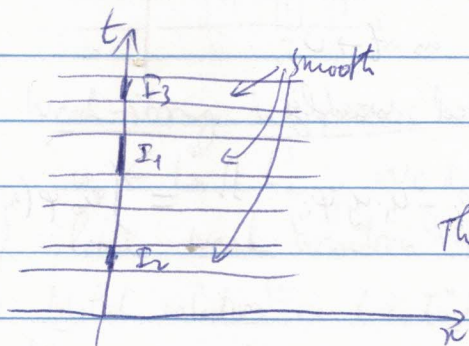
$$\bigcup_{I \in F_1^S} I \supset S.$$



$$\int_{I_1} f dt \geq \int_{t_1}^{t_1 + \epsilon} \frac{\epsilon}{\sqrt{t_1 - t}} dt \sim \epsilon \sqrt{\epsilon}$$

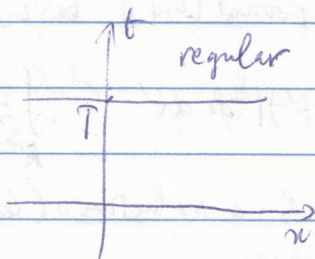
$$\sum_i \sqrt{\epsilon_i} < \frac{C}{\epsilon_1}$$

Then use the additional fact: $\forall \eta > 0, \exists \theta > 0: |E| < \theta \Rightarrow \int_E f < \eta$.
We can conclude that $\mathcal{H}^{1/2}(S) = 0$.



$$O = \mathbb{R}^3 \times \bigcup I_i$$

This result is by Leray.



We'll show that after some time T , the singularity no longer exists.

$$\int_0^\infty \int_{\mathbb{R}^3} |\nabla u(x,t)|^2 dx dt < \infty$$

$$\int_{\mathbb{R}^3} |u(x,t)|^2 dx \leq C$$

$\int |u(x,t)|^2 dx < \varepsilon_0 \Rightarrow$ the global solution exists

$$\left(\|u(\cdot, t_1)\|_{L^3} \leq \|u(\cdot, t_0)\|_{L^2}^{1/2} \|\nabla u(\cdot, t_0)\|_{L^2}^{1/2} \right)$$

Weak-strong uniqueness

1) Definition of weak solution of

$$\begin{cases} u_t + (u \cdot \nabla) u - \Delta u + \nabla p = 0 & \text{in } \mathbb{R}^n \\ u(x, 0) = u_0 \end{cases}$$

u is a weak solution if

(i) $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1$,

(ii) $u(\cdot, t) \rightarrow u_0$ in L^2 as $t \rightarrow 0$,

(iii) the equation is satisfied weakly:

$\forall \varphi$ div-free,

$$\int -u \varphi_t - u_i u_j \varphi_{ij} \dots = \int u_0 \varphi(x, 0) dx$$



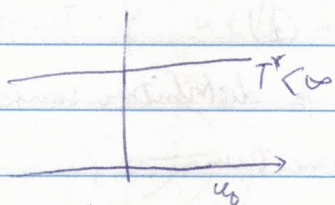
(iv) $t \mapsto u(\cdot, t)$ is continuous from $[0, T]$ to L_{weak}^2 .

(v) $\int -\frac{1}{2} |u(x,t)|^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u(x,t')|^2 dx dt' \leq \int_{\mathbb{R}^3} \frac{1}{2} |u_0(x)|^2 dx$

Note: we are able to construct such a solution (Galerkin's method which we discussed).

Theorem: (Leray, Serrin, ...)

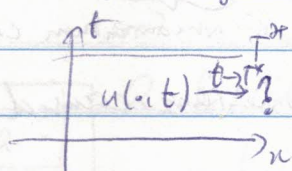
Assume $u_0 \in L^\infty \cap L^3$. Let $\bar{u}$ be a mild solution with $\bar{u}|_0 = u_0$.
 $(0, T^*)$ is the maximal interval of existence for the mild solution.
 Let u be a weak solution (in $\mathbb{R}^n \times (0, \infty)$).
 Consider $n=3$. Then $u = \bar{u}$ on $(0, T^*)$.



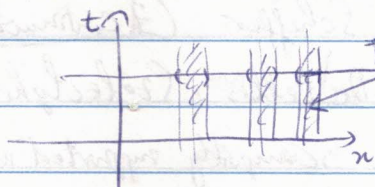
$\Rightarrow$ whenever a good solution exists, the weak solution must agree with it.

After time T^* , the model loses uniqueness $\Rightarrow$ lose predictive power.

Partial regularity in X .



$\exists S \subset \mathbb{R}^3$, $\bar{S} = S$ and $\mathcal{H}^1(S) = 0$



u is analytic when $t < T^*$

09/18/2014

Weak-strong uniqueness

$u_0 \in L^\infty \cap L^3$. We know:

- 1) global weak solution^(u) + energy inequality* (in the below sense)
- 2) Mild solution^(u) on $(0, T^*)$ - maximal time interval of existence

*: for a.e. t_1 and for $t_1 = 0$, we have

$$\forall t_2 \geq t_1, \int \frac{1}{2} |u(x, t_2)|^2 dx + \int_{t_1}^{t_2} \int |\nabla u(x, s)|^2 ds \leq \int \frac{1}{2} |u(x, t_1)|^2 dx$$

then $u = \bar{u}$ on $(0, T^*)$

Warning: if $u \in L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$ and equations satisfied in distribution, we don't know anything about this solution (energy inequality...)

This is an open problem. The weak solution in the weak-strong uniqueness ~~solo~~ theorem, the ~~solo~~ weak solution is the one we constructed by Galerkin's method.

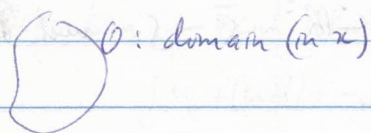
Example $u_t + (u \cdot \nabla)u + \nabla p = 0$

$$u_t + \operatorname{div}(u \otimes u) + \nabla p = 0 \quad (1)$$

u, p continuous: (1) makes sense in distribution sense.

This is the law of change of momentum:

$$\frac{d}{dt} \int_{\Omega} u(x,t) dx = - \underbrace{\int_{\partial\Omega} (u \cdot n)u}_{\text{momentum caused by movement of particles}} - \underbrace{\int_{\partial\Omega} p \cdot n}_{\text{momentum caused by forces}}$$



Ω : domain in $\mathbb{R}^N$

momentum caused by forces
momentum caused by movement of particles

"anomalous solution": 1993 Scheffer ($\exists$ continuous u)

2012 De Lellis, Szecelyhodi ($\exists u \in C^0$,
compactly supported in ^{space-time} ~~compactly~~)

The theorem of weak-strong uniqueness was stated in various forms

Leray: 1934

Serrin: 1960

Lenzi: 2000s

all ^{had to} imposed some energy estimate.

Connection to partial regularity

u : weak solution

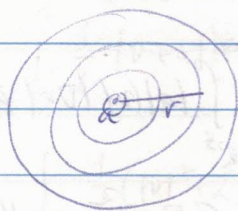
a.e. $\bar{t}$, $\int |\nabla u(\bar{t})|^2 dx < \infty$

$\int |u(\bar{t})|^2 dx \leq C < \infty$

$$u(\cdot, \bar{t}) \in L^2_x \cap L^6_x.$$

$u(x, t)$: mild solution starting at $u(\cdot, \bar{t})$ $\uparrow$
 $\xleftarrow{\quad \bar{t} \quad}$
 $u(\cdot, \bar{t})$

Recall: if $u \in H^1$ and $\Delta u = 0$



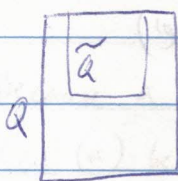
$$u|_{\partial B_{a,r}} \in H^1 \Rightarrow \underline{u} \in H^{3/2}.$$

solution to Poisson's equation

given by a Poisson integral. Then

we continue to gain regularity by $1/2$ each step.

$\Rightarrow u$ must be smooth.



$$Q = B_1 \times (-1, 0)$$

$$\int (|u(x, t)|^3 + |p(x, t)|^{3/2}) dx dt \leq \varepsilon_1 \quad (1980s: Schaffer)$$

$$\Rightarrow u|_{\partial \tilde{Q}} \text{ is small in } L^3$$

$$\Rightarrow u \in \text{better space (like in elliptic case)}$$

Need the weak-strong uniqueness theorem.

Sketch of proof:

a is a "smooth solution"

$u = a + v$ is another solution

$$v = u - a \Rightarrow u(x, 0) = 0$$

$$v_t + \underbrace{(a \cdot \nabla) v}_{\{1\}} + \underbrace{(v \cdot \nabla) a}_{\{2\}} + \underbrace{(v \cdot \nabla) v}_{\{3\}} - \underbrace{\Delta v}_{\{4\}} + \underbrace{v g}_{\{5\}} = 0. \quad (1)$$

$$\{1\} = \int g v_{ij} v_i = \int g \left(\frac{|v|^2}{2} \right)_{,ij} = 0$$

$$\{2\} = \int v_j a_{ij} v_i$$

$$\{3\} = \int v_j v_{ij} v_i = 0$$

$$\{4\} = \int -\Delta v \cdot v = \int |\nabla v|^2$$

$$\{5\} = \int \nabla q \cdot v = 0$$

then (1) becomes $\int_{\mathbb{R}^3} \frac{v^2(x,t)}{2} + \int_0^t \int_{\mathbb{R}^3} |\nabla v|^2(x,t') dx dt'$

$$\leq \int_{\mathbb{R}^3} |a| |v| |\nabla v| dx dt$$

$$\leq (\|v\|_{[0,t]}^2 \|a\|_{L_{t,x}^{n+2}(\mathbb{R}^d \times \mathbb{R}^3)})$$

Taking sup over t , we have something like

$$\|v\|_{[0,t]}^2 \leq \tilde{C} \|v\|_{[0,t]}^2 \|a\|_{L_{t,x}^{n+2}(\mathbb{R}^d \times \mathbb{R}^3)}$$

For t small, $\tilde{C} \|a\|_{L_{t,x}^{n+2}(\mathbb{R}^d \times \mathbb{R}^3)} < 1$

————— Then repeat the procedure in steps of time.

If v is not smooth, then $\{1\}, \{2\}, \{4\}, \{5\}$ are still dealt the same way. But we cannot do $\{3\}$ anymore because the integrand may not be integrable. The fact that u satisfies the energy estimate is essential for doing $\{3\}$.

04/24/2014

Weak-strong uniqueness

a is a regular solution (mild solution, which is smooth except at the initial time)

$$a(x,0) = a_0,$$

$$u = v + a \text{ weak solution.}$$

How to show that v satisfy the inequality

$$\int_{\mathbb{R}^3} \frac{v^2(x,t)}{2} + \int_0^t \int_{\mathbb{R}^3} |\nabla v|^2(x,t') dx dt' \leq \int_{\mathbb{R}^3} |a| |v| |v_0| dx dt?$$

The problem is that the term $\int v_i v_{ij} v_j$ may not be well-defined. Assume that u satisfies the energy inequality. (Otherwise, we don't know anything...)

$$\begin{aligned} \int_{at}^t \frac{1}{2} |v+a|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla(v+a)|^2 dx dt' &\leq \frac{1}{2} \int_{\mathbb{R}^3} |v_0|^2 dx \\ \Rightarrow \int_{at}^t \frac{1}{2} (|v|^2 + 2va + |a|^2) dx + \int_0^t \int_{\mathbb{R}^3} (|\nabla v|^2 + 2\nabla v \cdot \nabla a + |\nabla a|^2) dx dt' \\ &\leq \frac{1}{2} \int_{\mathbb{R}^3} |v_0|^2 dx \quad (*) \end{aligned}$$

Because a is a regular solution, it satisfies the energy identity.

then (*) becomes $\int_{at}^t \left(\frac{1}{2} |v|^2 + va \right) dx + \int_0^t \int_{\mathbb{R}^3} (|\nabla v|^2 + 2\nabla v \cdot \nabla a) dx dt' \leq 0$ (1)

We have $\frac{\partial}{\partial t} \int v(x,t) a(x,t) dx = \int (v_t a + v a_t) dx$ (**)

Because $u = v+a$ satisfies $u_t + (u \cdot \nabla) u - \Delta u + \nabla p = 0$ in sense of distribution, we have

$$v_t + a \nabla v + v \cdot \nabla a + (v \cdot \nabla) v - \Delta v + \nabla q = 0. \quad (***)$$

$$\text{RHS}(**) = \int_{at}^t [a(-a \cdot \nabla v - v \cdot \nabla a - v \cdot \nabla v + \Delta v) + v(-a \cdot \nabla a + \Delta a)] dx$$

Note that we never have the term $v_i v_{ij} v_j$. Where did it go? Secretly, it disappears when we assume that u satisfies the energy inequality.

From (**), we formally have

$$\partial_t \int \frac{|v|^2}{2} + \int v \nabla a \cdot v + \int |vv|^2 = 0 \quad (2)$$

(1) is essentially equivalent to (2).

As a consequence, (of weak-strong uniqueness theorem)

a weak solution + energy inequality + $(u(x,0)=0) \Rightarrow u \equiv 0$.

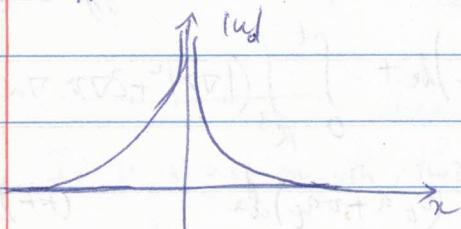
(because the energy has to be zero at all time)

$\Rightarrow$ the energy inequality is a strong assumption.

Question

Uniqueness of weak solution with energy inequality?
(for $u_0 \in L^\infty$)

This is an open problem. Conjecture (Hao-Jia): $\exists u_0$ compactly supported, smooth in $\mathbb{R}^1 \setminus \{0\}$, $|u(x)| \sim \frac{C}{|x|}$ at 0.



There are infinitely many weak solutions starting with energy inequality starting from u_0 .

$$\begin{aligned} \exists (u_0^{(n)}) &\rightarrow u_0, \quad u^n \rightarrow u \\ (\tilde{u}_0^{(n)}) &\rightarrow u_0, \quad \tilde{u}^n \rightarrow \tilde{u} \end{aligned}$$

Note that if $|u_0| \sim \frac{C}{|x|^{1-s}}$ around 0 then the previous problem never happens because the weak-strong uniqueness theorem applies.

Methods $\begin{cases} \rightarrow \text{perturbation theory} \\ \rightarrow \text{energy inequality.} \end{cases}$

Note: if u_0 is smooth, then the weak solution with energy inequality is unique.

04/23/2014

In 1980s, people studied the well-posedness of NSE: $u_0 \in X$

$$X = \dot{H}^{1/2}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$$

$$X = L^3(\mathbb{R}^3)$$

$$X = B_{p,1}^s \dots \text{ (Besov space)}$$

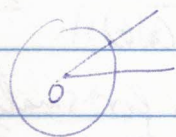
$$\begin{cases} u_t + \dots = 0 \\ \operatorname{div} u = 0 \\ u(x,0) = 0 \end{cases}$$

Scaling $u(x,t) \rightarrow u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$

$$u_0(x) \rightarrow u_\lambda(x) = \lambda u_0(\lambda x)$$

u_0 is "scale-invariant" $\Leftrightarrow u_0$ is (-1) -homogeneous, i.e.

$$u_0(x) = A \left(\frac{x}{|x|} \right) \frac{1}{|x|}$$



One way to produce such a u_0 is to take any function B homogeneous ($B(x) = B(\frac{x}{|x|})$), and then take

$$u_0 = \operatorname{curl} B$$

$$\frac{1}{|x|} \text{ "just misses" } L^3; \quad \frac{1}{|x|} \notin L^3(\mathbb{R}^3)$$

$$\frac{1}{|x|} \in L^3_{\text{weak}} = L^{\frac{3}{2}}; \quad \frac{1}{|x|} \in B_{p,q}^s \text{ (suitable choice).}$$

u_0 scale-invariant data produces scale-invariant solution u .

u_0 is rotational invariant, i.e. $u_0(Rx) = Ru_0(x)$



"axis-symmetric"



$$u = u^{(r)}(r, z) e_r + u^{(\theta)}(r, z) e_\theta + u^{(z)}(r, z) e_z$$

If u_0 is axis-symmetric and no-swirl (i.e. $u^{(\theta)} = 0$) then $\exists$ a global regular solution.

Note that the solutions may not be axis-symmetric (for weak solutions) due to the nonuniqueness. But for strong solutions, they are still axis-symmetric.

Question: Can we find some function space containing (-1)-homogeneous function and satisfying the local existence by perturbation theory?

$$\left. \begin{array}{l} T \\ u \text{ scale-invariant} \\ u_0 \text{ scale-invariant} \end{array} \right\} \Rightarrow T = \infty$$

At that time, we will have a global solution, because

$$\lambda u(\lambda x, \lambda^2 t) \mapsto u(x, t)$$

* Elementary proof of well-posedness for small (-1)-homogeneous data:

$$X = \left\{ u_0: \mathbb{R}^3 \rightarrow \mathbb{R}^3, \operatorname{div} u_0 = 0, |u_0(x)|/|x| \text{ is bounded, } u_0 \text{ is continuous in } \mathbb{R}^3 \setminus \{0\} \right\}$$

$$\|u_0\|_X := \sup \{ |x| |u_0(x)| \} = \|x u_0\|_{L^\infty} \quad \rightarrow \text{or } L^\infty$$

Where do the solutions live?

$$u_0 \in X \Rightarrow \underbrace{\Gamma(t) * u_0}_{\text{heat kernel}} \in ?$$

Recall: if $X = L^p_x$ then $X = L^5_{t,x}$ (simplest choice among many)

If $u_0 \sim \frac{1}{|x|}$, what does the solution look like?

$$u(x, t) \sim \frac{1}{\sqrt{t+|x|}} \sim \frac{1}{(t+|x|)^{1/2}}$$

$$X := \{ u_0: \sup (\sqrt{t+|x|}) |u| < \infty \}$$

$$\|u\|_X := \sup(\sqrt{t} + |x|)|u|$$

Now we'll show that this space satisfies the perturbation theory.

$$u \in X$$

$$\|B(u,v)\|_X \stackrel{?}{\leq} C\|u\|_X\|v\|_X$$

$$u_0 \in X$$

$$\begin{cases} w_t - \Delta w + \nabla q = \pm \mathcal{B}(u,v) \\ \operatorname{div} w = 0 \\ w(t_0) = 0 \end{cases}$$

$$w(t) = \int_0^t K'(s) * (uv)(t-s) ds$$

We can assume wlog (scaling) that $\|u\|_X = \|v\|_X = 1$.

$$|w(t)| \leq \frac{C}{|x| + \sqrt{t}} \quad ; \quad |K'(y,s)| \leq \frac{C}{(|y|^2 + s)^{\frac{n+1}{2}}}$$

$$|u||v| \leq \frac{1}{|x| + \sqrt{t}}$$

$$\Rightarrow |w| \lesssim \frac{1}{|x|^2 + t}$$

Thus, we want to show $\int_0^t \int_{\mathbb{R}^n} \frac{1}{(s+|y|^2)^{\frac{n+1}{2}}} \frac{1}{(t-s)+|y-x|^2} dy ds \leq \frac{C}{|x| + \sqrt{t}}$

This is a nontrivial exercise but not terribly hard.

$$u \in X$$

heat extension

$$u_0 \in X$$

Theorem: if $\|u_0\|_X \leq \varepsilon$, then $\exists!$ scale invariant $u \in X$.

The scale-invariance comes from the iteration procedure (we never leave the space of $(+)$ -homogeneous, so the limit must live there also).

Question: Last time we called a function u_0 scale-invariant if it is (-1) -homogeneous. We also stated a theorem saying if u_0 is scale-invariant and $\|u_0\|_X$ is small enough then there exists a global scale-invariant solution $u \in X$. The question is: for a function u depending on t and x , what does scale-invariance mean?

04/25/2014

Answer the question:

u is scale-invariant $\stackrel{\text{def}}{\iff} \lambda u(\lambda x, \lambda^2 t) = u(x, t) \quad \forall \lambda > 0$

Equivalently, $u(x, t) = \frac{1}{\sqrt{t}} u\left(\frac{x}{\sqrt{t}}, 1\right) = \frac{1}{\sqrt{t}} U\left(\frac{x}{\sqrt{t}}\right)$

$\|u\|_X$ small $\Rightarrow$ ^{global} scale-invariant u exists.

Large data?

Remark: $X, \mathcal{X}$
 $u_0 \hookrightarrow U$ - heat extension

$u = U + B(u, u)$ - equation in X .

- equation in $\mathcal{X}_T$ - functions on $\mathbb{R}^n \times [0, T]$.

Recall: in L^n -theory, $\|U\|_{\mathcal{X}_T} \rightarrow 0$ as $T \rightarrow 0^+$

This is, however, not true for our current setting

$\mathcal{X}_T = \{u: \mathbb{R}^3 \times (0, T) \rightarrow \mathbb{R}^3, \sup(\sqrt{t} + |x|)|u| < \infty\}$.

Thus, we cannot have local existence for large data. We only have global existence for large data. For large data, we need to go beyond the perturbation theory.

$$\frac{1}{|x|} \in X, \quad \frac{1}{|x| + \sqrt{t}} \in \mathcal{X}$$

$L^3 \not\subset \frac{1}{|x|}$ local-in-time existence, large data

$\left. \begin{matrix} L^3 \\ \text{weak} \\ \times \end{matrix} \right\} \Rightarrow \frac{1}{|x|}$ global-in-time existence, small data.

$$u(x,t) = \frac{1}{\sqrt{t}} U\left(\frac{x}{\sqrt{t}}\right), \quad U: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

1) NSE for u gives:

$$-\Delta U - \frac{1}{2} x \cdot \nabla U - \frac{1}{2} U + (U \cdot \nabla) U + \nabla P = 0$$

$$\operatorname{div} U = 0$$

2) $u(x,0) = u_0$ becomes $U(x) = u_0(x) + o\left(\frac{1}{|x|^2}\right), |x| \rightarrow \infty$

$$U(x) = \frac{1}{\sqrt{t}} u_0\left(\frac{x}{\sqrt{t}}\right) + \frac{1}{\sqrt{t}} \left(\frac{\sqrt{t}}{|x|}\right)^2 + \dots$$

For U , we basically have an elliptic equation



" $\partial \mathbb{R}^3$ "

After mapping $\mathbb{R}^3$ into a ball, we get a boundary condition on the ball.

$$\begin{cases} -\Delta u + (u \cdot \nabla) u + \nabla p = 0 & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u|_{\partial \Omega} = g \end{cases} \quad \text{Model 1}$$

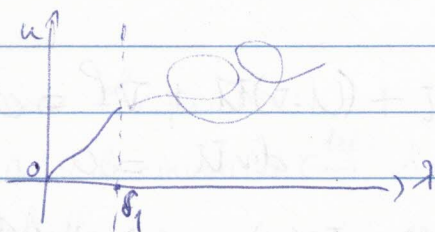
$$\begin{cases} -\Delta u + (u \cdot \nabla) u + \nabla p = f & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \\ u|_{\partial \Omega} = 0 \end{cases} \quad \text{Model 2}$$

Model 1 is still an open problem for complicated domain Ω .

Model 2 is more convenient.

Model 2: (i) small p : perturbation, implicit function
 (ii) large p : "fixed point; topological degree theory"

$$\begin{cases} -\Delta u + (u \cdot \nabla) u + \nabla p = \lambda f \\ \operatorname{div} u = 0 \\ u|_{\partial \Omega} = 0 \end{cases}$$



$$\lambda = 0 \Rightarrow u \equiv 0$$

Think of finite-dimensional picture $u = (u_1, \dots, u_m) \in \mathbb{R}^m$, $m \gg 1$
 (truncation of solution in some finite-dim space)

$$F(u, \lambda), F: \mathbb{R}^m \times [0, 1] \rightarrow \mathbb{R}^m$$

$$F(u, \lambda) = 0$$

$$F(u, 0) = u$$

$\{(u, \lambda): F(u, \lambda) = 0\}$ may look irregular even if F is smooth.

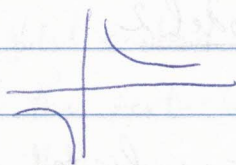
Sard's theorem:

$F: \mathbb{R}^N \rightarrow \mathbb{R}^k$ is smooth

$\{x: F(x) = y\}$ may not be a smooth manifold.

Sard's theorem says that it is a smooth manifold for almost every $y \in \mathbb{R}^k$.

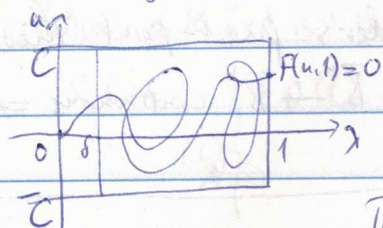
Ex: $F: \mathbb{R}^2 \rightarrow \mathbb{R}$, $F(x) = x_1 x_2$



$\{x: F(x) = 0\}$ is not a smooth manifold

Thus, we can assume $\{(u, \lambda): F(u, \lambda) = y\}$ is a smooth manifold for

γ arbitrarily small.



The curve cannot return to the region $\{(u, \lambda): 0 \leq \lambda < \delta\}$ because there we have the uniqueness.

The curve cannot wind infinitely frequently in a compact set (otherwise, it will cease to be a manifold)

$\Rightarrow$ it must reach $\lambda = 1$

$\Rightarrow$ Model 2 must have a solution.

What need to show is an a priori estimate:

$\forall \lambda \in [0, 1], \forall \text{ solution } u, F(u, \lambda) = 0$, we have $|u| \leq C$.

Trouble: Sard's theorem is true only for finite-dim space

$$\begin{cases} -\Delta u + (u \cdot \nabla) u + \nabla p = \lambda f & (\text{fixed } \lambda) \\ u|_{\partial \Omega} = 0 \\ \operatorname{div} u = 0 \end{cases} \quad (1)$$

We'll show $\forall \lambda \in [0, 1], \|\nabla u\|_2 \leq C_f$.

$$\begin{aligned} \text{Proof: } (1) \Rightarrow \int_{\Omega} |\nabla u|^2 &= \int_{\Omega} \lambda f u \leq \lambda \|f\|_2 \|u\|_2 \\ &\leq C \lambda \|f\|_2 \|\nabla u\|_2 \end{aligned}$$

$$\Rightarrow \|\nabla u\|_2 \leq C \|f\|_2 \quad \forall [0, 1] \ni \lambda.$$

04/28/2014

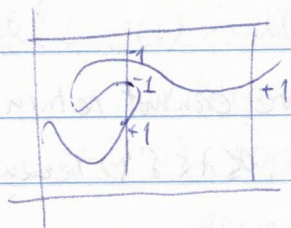
No Class on Wednesday. No office hour today.

New homework assignment handed out.

Continue the discussion last time:

$$F(u, \lambda) = 0$$

1) "degree theory": Leray-Schauder degree.



degree = number of zeros counted with signs

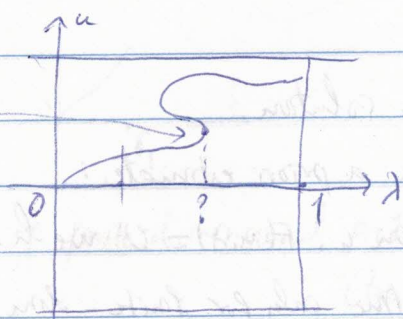
* Browder's fixed-point theorem:

$$\left\{ \begin{array}{l} f: \bar{B}^n \rightarrow \bar{B}^n \text{ continuous} \Rightarrow \exists \bar{x} \in \bar{B}^n: \\ f(\bar{x}) = \bar{x} \end{array} \right.$$

"Proof": $F(x, \lambda) = (1-\lambda)x + \lambda f$

Then use Sard's theorem.

"Saddle-node"



$$F(u(\lambda), \lambda) = 0$$

$$F(u(\lambda)/v, \lambda) = 0$$

$$v \rightarrow F_u(u(\lambda), v) \cdot v$$

$L_{u(\lambda)}$ linearized operator.

Spectrum of $L_{u(\lambda)}$ can tell about

where the saddle-node is: the point on the real line passes

from the left to the right of 0 as we move on the curve passing the saddle-node

$$\begin{array}{l} \therefore \text{small } \lambda \\ \text{Re } \lambda \leq 0 \end{array}$$

$$\left\{ \begin{array}{l} -\Delta u - \frac{1}{2} x \cdot \nabla u - \frac{1}{2} u + (u \cdot \nabla) u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u(x) = \underbrace{u_0(x)}_{(-1)\text{-homogeneous}} + O\left(\frac{1}{|x|}\right) \end{array} \right.$$

as $x \rightarrow \infty$

$$\text{A-priori estimate: } \int_{\mathbb{R}^n} |\nabla u|^2 \leq C \quad (*)$$

Mild-solution: $u \in L_{loc}^5 \Rightarrow u$ is as smooth as Calderon functions (solution to heat equation)

"bootstrapping".

signs

For steady state 3D, (*) controls regularity: any weak solution with (*) is smooth.

$\bar{x}$

$$\text{Return: } \begin{cases} -\Delta U - \frac{1}{2} x \cdot \nabla U - \frac{1}{2} U + (U \cdot \nabla) U + \nabla P = 0 \\ \text{div } U = 0 \end{cases}$$

$$U(x) = u_0(x) + o\left(\frac{1}{|x|}\right) \quad (x \rightarrow \infty)$$

How to find a space X for the topological argument (such that each solution to (I_2) satisfies (*)).

It is hard to do so from $-\Delta U - \frac{1}{2} x \cdot \nabla U - \frac{1}{2} U + (U \cdot \nabla) U + \nabla P = 0$.
We need to return to the time-dependent model (the one before assuming self-similarity).

$$u(x,t) = \frac{1}{\sqrt{t}} \bar{U}\left(\frac{x}{\sqrt{t}}\right)$$

For $n=1$:
$$U(y) = \frac{a_1}{y} + \frac{a_2}{y^2} + \frac{a_3}{y^3} + \dots$$

$$\begin{aligned} \Rightarrow \frac{1}{\sqrt{t}} \bar{U}\left(\frac{x}{\sqrt{t}}\right) &= \frac{1}{\sqrt{t}} \left(\frac{a_1 \sqrt{t}}{x} \right) + \frac{1}{\sqrt{t}} \frac{a_2 t}{x^2} + \frac{1}{\sqrt{t}} \frac{a_3 t^{3/2}}{x^3} + \dots \\ &= \frac{a_1}{x} + \sqrt{t} \frac{a_2}{x^2} + t \frac{a_3}{x^3} + t^{3/2} \frac{a_4}{x^4} + \dots \end{aligned}$$

To be in the perturbative theory, $u(x,t)$ should behave like the solution to heat equation $\Rightarrow$ is very smooth near $t=0$.

$\Rightarrow a_2 = a_4 = \dots = 0$ should be assumed

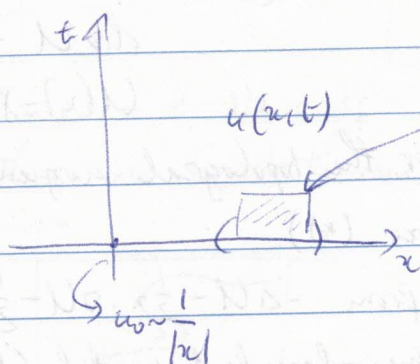
$$\Rightarrow U(y) = u_0(y) + o\left(\frac{1}{|y|^3}\right)$$

function

A-priori estimate:
$$|U - u_0| \leq \frac{C}{|x|^3} \quad \text{at } \infty.$$

This a-priori estimate will replace the energy estimate.
 When we have this estimate, we can use the degree theory.

$$|x|^3 |U - \lambda u_0| \leq C \quad \forall x \geq |R_0|$$



u "sufficiently regular"

This regularity seems to be obvious in heat equation, but not obvious for elliptic equation.

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$$\begin{cases} -\Delta U - \frac{1}{2}x \nabla U - \frac{1}{2}U + U \nabla P + \nabla P = 0 \\ U(x) = u_0(x) + o\left(\frac{1}{|x|}\right) \end{cases}$$

Discuss the uniqueness / nonuniqueness today.

Strategy: add artificial time s .

$$\begin{cases} U_s - \Delta U - \frac{1}{2}x \nabla U - \frac{1}{2}U + U \nabla P + \nabla P = 0 \\ U(x) = u_0(x) + o\left(\frac{1}{|x|}\right) \text{ (uniformly in } s) \end{cases}$$

$$U = U(y, s), \quad s \in (-\infty, 0), \quad y \in \mathbb{R}^3$$

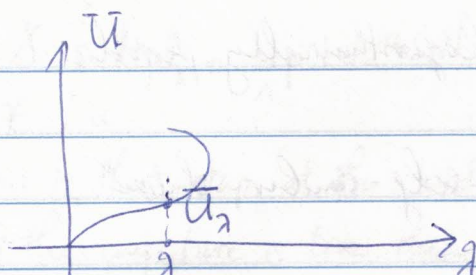
$$\text{Put } u(x, t) = \frac{1}{\sqrt{t}} U\left(\frac{x}{\sqrt{t}}, \frac{\log t}{2}\right) \quad \forall x \in \mathbb{R}^3, \forall t \in (0, 1)$$

Note: $t = e^s$ and $t^0 = e^{s^0}$.

Consider the problem

$$\begin{cases} U_s - \Delta U - \frac{1}{2}x \nabla U - \frac{1}{2}U + U \nabla P + \nabla P = 0 \\ U(x) = \lambda u_0(x) + o\left(\frac{1}{|x|}\right) \end{cases}$$

small λ



$$U(y,s) = \bar{U}_\lambda + \underbrace{V}_{\text{"small"}}$$

(linear approximation)

$$V_s = \mathcal{L}_\lambda V \quad (*)$$

spectrum

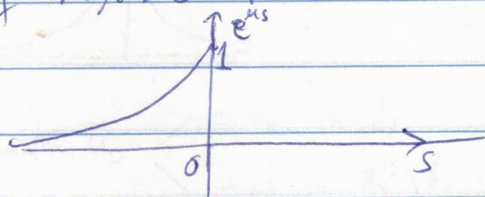
$$\mathcal{L}_\lambda V = \Delta V + \frac{1}{2} \kappa V + \frac{1}{2} V - \bar{U}_\lambda \cdot \nabla V - V \cdot \nabla \bar{U}_\lambda + \dots$$

Example:

$$\mathcal{L}_\lambda \phi = \underbrace{\mu \phi}_{\mu \in \mathbb{C}} \quad \phi: \text{decay, continuous}$$

The solution $V = \kappa e^{\mu s} \phi$ solves (*). (κ is a constant)

If $\operatorname{Re} \mu > 0$ then $e^{\mu s} \rightarrow 0$ as $s \rightarrow -\infty$.



$$U(y,s) = \bar{U}_\lambda(y) + \kappa e^{\mu s} \phi(y) + O(\kappa^2).$$

$$\Rightarrow u(x,t) = \underbrace{\frac{1}{\sqrt{t}} \bar{U}_\lambda\left(\frac{x}{\sqrt{t}}\right)}_{\bar{u}(x,t)} + \underbrace{\kappa t^\mu \phi\left(\frac{x}{\sqrt{t}}\right)}_{\text{if } \operatorname{Re}(\mu) > 0, \text{ this term} \rightarrow 0 \text{ as } t \rightarrow 0} + O(\kappa^2).$$

$\bar{u}(x,t) \rightarrow u_0(x)$ as $t \rightarrow 0$

$\Rightarrow$ the function $\frac{1}{\sqrt{t}} \bar{U}_\lambda\left(\frac{x}{\sqrt{t}}\right) + \left(\frac{\kappa}{\sqrt{t}}\right) \phi\left(\frac{x}{\sqrt{t}}\right)$ satisfies the same

initial condition $\Rightarrow$ nonuniqueness.

$$\mu = \varepsilon + i\eta$$

$$\Rightarrow t^{\varepsilon+i\eta} = t^\varepsilon e^{i\eta \log t}$$

$$\Rightarrow \operatorname{Re}(t^{\varepsilon+i\eta}) = t^\varepsilon \cos(\eta \log t).$$



logarithmically periodic

→ "discretely self-similar solution".

Explain the above term:

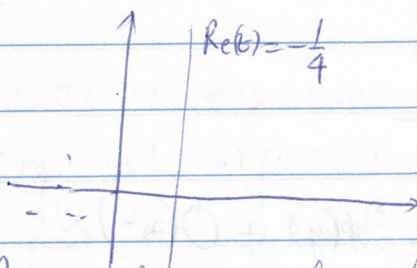
Scale-invariant means: $\lambda u(\lambda x, \lambda^2 t) = u(x, t) \quad \forall \lambda > 0$

discretely self-similar means: $\lambda u(\lambda x, \lambda^2 t) = u(x, t) \quad \forall \lambda \in \{\lambda_0, \lambda_0^2, \lambda_0^3, \dots\}$
where λ_0 depends on η (simple calculation).

Summarize:

for each $\lambda > 0$, we have the problem corresponding to λu_0 ; then get a linearized U_λ solution.

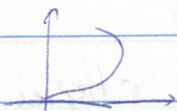
$\mathcal{L}_\lambda \rightarrow$ spectrum.



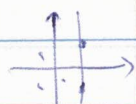
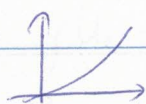
$\lambda = 0$: all eigenvalues have negative real parts.

Scenario:

① "saddle point": the eigenvalues have real parts crossing 0 (become positive)



② "second discretely self-similar solution": nothing weird happens but when we perturb the solution, the instability occurs.



Hopf bifurcation

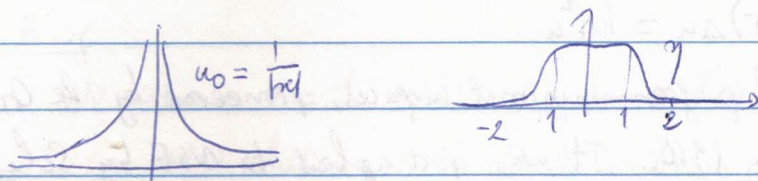
Whenever Real part of the eigenvalue(s) > 0 , nonuniqueness occurs.

$$X, \|u_0\|_X = \sup_X |x| |u_0(x)|$$

This is the setup at the borderline where the perturbation theory.
If the conjecture is true then there'll be a huge difference between L^3 and L^3_{weak} , because

$$L^3 \subsetneq L^3_{weak}$$

Theorem non-unique solutions with (-1) -homogeneous data u_0 can be truncated at $|x| > 1$.

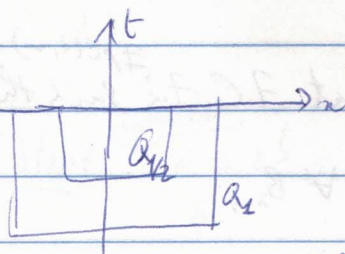


Truncation $\bar{u}_0 = \begin{cases} u_0 & \text{in } B_1 \\ 0 & \text{in } \mathbb{R}^3 \setminus B_2 \end{cases}$ still nonuniqueness.

ϵ -regularity lemma

u, p weak solution + energy inequality

$$\exists \epsilon > 0: \int_{Q_1} (|u|^3 + |p|^{3/2}) dx dt < \epsilon \Rightarrow (u, p) \text{ is regular in } Q_{1/2}$$



This is hard to prove. It contains the pressure $\Rightarrow$ complicated (not distracting from the essence of the problem)

Model problems

(i) 5d - steady NSE: still not so good because it still contains the pressure.

(80) 5d- steady GGL (Ginzburg-Landau)

$$-(1-i\sigma)\Delta u = |u|^2 u \quad \sigma > 0$$

where $u: \Omega \subset \mathbb{R}^5 \rightarrow \mathbb{C}$.

It is known that this equation has a singular solution.

$$u = \frac{A}{r^{1+i\sigma}}$$

ε -regularity in B_1 : if $\int_{B_1} |u|^3 < \varepsilon$ then u is smooth in $B_{1/2}$.

05/05/2014 Model equation:

$$-(1-i\sigma)\Delta u = |u|^2 u$$

This model is from minimal surfaces, pioneered by De Giorgi in late 1950s and early 1960s. It was first applied to NSE by Scheffer in 1970.

① Campanato's characterization of C^α . (see the book by E. Giusti in the library).

Consider $f \in C^\alpha(\mathbb{R}^n)$. We want to characterize the Hölder continuity by integral.

$$f \in C^\alpha(\mathbb{R}^n) \Rightarrow \int_B \underbrace{\left| \frac{f(x) - f(y)}{|x-y|^\alpha} \right|^p}_{\sim r^\alpha} dx \leq C r^{\alpha p}$$

Campanato: If $f \in L^1_{loc}(\mathbb{R}^n)$ such that $\exists C, \exists \alpha$ such that $\exists p \in (1, \infty)$

$$\int_B \left| \frac{f(x) - f(y)}{|x-y|^\alpha} \right|^p dx \leq C r^{\alpha p} \quad \forall B_{x,r}$$

Then $f \in C^\alpha$ and $\frac{|f(x) - f(y)|}{|x-y|^\alpha} \leq A$ where $\alpha = \alpha - \frac{n}{p} \in (0,1)$
A depends only on α, p, n

This is much simpler than BMO. Note that p doesn't play a big role.

Campanato's theorem is actually stronger than Sobolev's imbedding $W^{1,p} \hookrightarrow C^\alpha$ ($p > n$) but the proofs are similar.

ε -regularity lemma ($n=5$)

$$\frac{1}{r^2} \int_{B_{nr}} |u - (u)_{B_{nr}}|^3 dy \quad I(nr) = \frac{1}{r^2} \int_{B_{nr}} |u(y) - (u)_{B_{nr}}|^3 dy$$

(scale-invariant)

If $\exists \varepsilon_0 > 0$ such that $I(nr) < \varepsilon_0$ then u is smooth in B_{nr} .

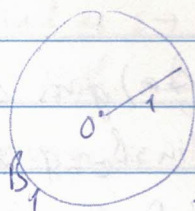
Moreover, for each $\tau \in (0, \frac{1}{2})$, $\exists \varepsilon = \varepsilon(\tau)$, $\exists C$ s.t. if $I(nr) < \frac{\varepsilon}{r^3}$ then $I(n\tau r) \leq C\tau^6 I(nr)$.

Proof

We assume $r=1$ WLOG (by rescaled shift to 0 and then rescale).

$u(x)$ solution $\Rightarrow \lambda u(\lambda x)$ is also a solution.

Put $a = \int_{B_1} u$. Write $u = a + \varepsilon v$.



$$\int_{B_1} |v|^3 = 1, \quad \int v = 0$$

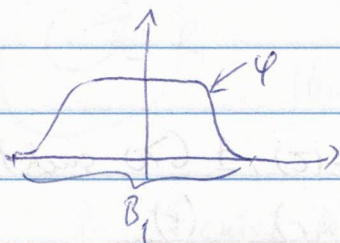
The equation is $-(1-\varepsilon)\Delta(a + \varepsilon v) = |a + \varepsilon v|^2(a + \varepsilon v)$.

Since a is constant, we get

$$-(1-\varepsilon)\varepsilon\Delta v = |a|^2 a + \varepsilon P(a, v) + \varepsilon^2 Q(a, v) + \varepsilon^3 |v|^2 v + \varepsilon^3 |v|^2 v$$

quadratic
linear
linear
quadratic

Step 1 ^{consider} $a = \varepsilon b$, $|b| \leq C$



$$\int_{B_1} \varphi = 1 \quad \varphi \text{ is a cutoff function}$$

Multiply both sides of (*) by φ and take the integral.

$$\varepsilon \underbrace{\int_{B_1} (1-i\sigma) v \Delta \varphi dx}_{O(1)} = \underbrace{\int_{B_1} |a|^2 a \varphi}_{|a|^2 a} + \dots$$

$$\text{Then } \varepsilon O(1) = |a|^2 a + \varepsilon O(1) |a|^2 + \varepsilon^2 |a| O(1) + \varepsilon^3 O(1)$$

$$\Rightarrow \frac{\varepsilon O(1)}{|a|^2} = a + \varepsilon O(1) + \frac{\varepsilon^2}{|a|} O(1) + \varepsilon^3 \frac{O(1)}{|a|^2}$$

$$\Rightarrow a = \underbrace{\varepsilon b}_{\text{bounded, independent of } \varepsilon}$$

$$\Rightarrow -(1-i\sigma) \Delta v = \varepsilon^2 \underbrace{P(b, v)}_{\text{bounded}} \quad (**)$$

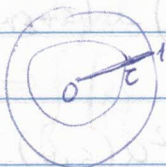
For small ε , the inequality to be shown is that $\exists C$:

$$\int_{B_\varepsilon} |v - (v)_{B_\varepsilon}|^3 \leq C \varepsilon^3$$

The idea is to look at the case $\varepsilon \geq 0$: $(**)$ gives $\Delta v = 0$.

When v is harmonic, we control everything in B_ε if we know

$$v \text{ in } B_1 \quad \text{we know that } \int_{B_1} |v|^3 dx = 1.$$



We can control Lipschitz-continuity constant:

The claim of the theorem is basically the perturbation theory.

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$$-(1-i\sigma) \Delta u = |u|^2 u.$$

$$-(1-i\sigma) \Delta \varphi = \dots + \underbrace{\varepsilon^4 |u|^2 v}_{\text{bad term}}$$

$\exists \varepsilon > 0$ Given $\tau \in (0, 1)$, $\exists \varepsilon > 0$, $\varepsilon = \varepsilon(\tau)$, $\exists C > 0$ such that if $I(x, r) < \varepsilon$ then $I(x, \tau r) < C \tau^6 I(x, r)$. (*)

Suppose by contradiction that this is false, then there exists a sequence $\varepsilon_n \downarrow 0$ and

$$\begin{cases} -(1-i\varepsilon) \Delta v_\varepsilon = \varepsilon^2 \underbrace{(\dots)}_{\text{bounded in } L^{3/2}} + \varepsilon^2 |v_\varepsilon|^2 v_\varepsilon, \\ \int |v_\varepsilon|^3 = 1. \end{cases} \quad (I)$$

v_ε violates (*). We can assume $r=1$.

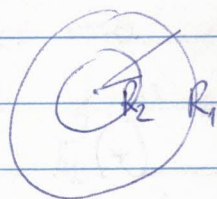
$$\int_{B_\varepsilon} |v_\varepsilon - (v_\varepsilon)_{B_\varepsilon}|^3 \geq C \varepsilon^6 \underbrace{\int_{B_1} |v_\varepsilon|^3}_{=1} \quad (**)$$

We know that this inequality if v_ε were harmonic (basic property) (v_ε) is bounded in L^3 , thus $v_\varepsilon \xrightarrow{L^3} v$ and $\Delta v = 0$, $\|v\|_3 \leq 1$.
Now the question is how to show that we can pass through the limit as $\varepsilon \rightarrow 0$ in (**). It suffices to show $v_\varepsilon \xrightarrow{L^3} v$ in $B_{1/2}$.

With only (I), we need to turn weak-convergence to strong convergence. We will need $\int_{B_{3/4}} |\nabla v_\varepsilon|^2 dx \leq C$

Then by Sobolev's imbedding we get the strong convergence. This is a general strategy for nonlinear PDE (Compactness theorem)

Local energy inequality
"Caccioppoli's inequality"



$$\Delta u = 0 \Rightarrow \int_{B_{R_2}} |\nabla u|^2 \leq \frac{C}{(R_1 - R_2)^2} \int_{B_{R_1}} |u|^2$$

How to prove?

$$\eta(x) = \begin{cases} 1, & |x| \leq R_2 \\ 0, & |x| \geq R_1 \end{cases}$$

smooth cutoff function.

$$0 = \int (\Delta u) u \eta^2 dx = \int |\nabla u|^2 \eta^2 dx \leq \int 2(u \eta) \cdot (u \cdot \nabla \eta)$$

$$\leq \varepsilon \int |\nabla u|^2 \eta^2 + \frac{\varepsilon}{\varepsilon} \int u^2 |\nabla \eta|^2$$

We'll manipulate the equation $-(1-i\sigma)\Delta u = |u|^2 u$. The same will be true for the equation $-(1-i\sigma)\Delta v_\varepsilon = \varepsilon^2(\dots) + \varepsilon^2 |v_\varepsilon|^2 v_\varepsilon$.

Multiply both sides of $-(1-i\sigma)\Delta u = |u|^2 u$ by $i u \eta^2$, we get

$$0 = \nabla[(1-i\sigma)u] \cdot \nabla(i u \eta^2) = \sum_{k=1}^3 (\dots)$$

$$(u = u_1 + i u_2)$$

Then we return to the Caccioppoli's inequality.

Def of weak solution:

$$(i) \int |\nabla u|^2 < \infty \quad (u \in H^1)$$

$$(ii) \int (1-i\sigma) \nabla u \cdot \nabla \varphi = \int |u|^2 u \varphi$$

Local energy inequality: $\varphi = i u \eta^2$.

$$H^1 \hookrightarrow L^6 \quad n=3$$

$$L^4 \quad n=4$$

$$L^{10/3} \quad n=5$$

Good exercise: every weak solution to $-(1-i\sigma)\Delta u = |u|^2 u$ is smooth for $n=4$.

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$$a_\varepsilon = \varepsilon b, \quad b \text{ bounded}$$

$$u_\varepsilon = \varepsilon b_\varepsilon + \varepsilon v_\varepsilon, \quad b_\varepsilon \rightarrow b$$

$$-(1-i\sigma)\Delta v_\varepsilon = \varepsilon^2(|b_\varepsilon|^2 b_\varepsilon + \dots + |v_\varepsilon|^2 v_\varepsilon) \quad (*)$$

$$\int_{B_1} |v_\varepsilon|^3 = 1, \quad v_\varepsilon \rightarrow v$$

$$-(1-\varepsilon)\Delta u = |u|^2 u$$

Multiply by $u\eta^2$ (η is a cutoff function) $\Rightarrow$ get ^{local} energy ^{inequality} identity.

We need to assume the weak solutions are in L^4 . This is not available for $n=5$ because $H^1 \hookrightarrow L^{10/3}$ ($u \in H^1 \not\Rightarrow u \in L^4$).

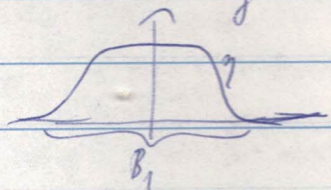
$$u: B_1^3 \rightarrow S^2, \quad -4u = |\nabla u|^2 u$$

Partial regularity: L.C. Evans (1990s) (weak solution + "stability")

weak solution $\nrightarrow$ partial regularity (T. Riviere, 1990?)

"suitable weak solution" = weak solution + local energy regularity.

(*) $\rightarrow$ "test" by multiplying $u\eta^2$, then get right hand side:



$$\left. \begin{aligned} &\sim |\nabla u|^2 \eta^2 \\ &|\nabla u|^2 \eta^2 \\ &\dots \end{aligned} \right\} \eta^2$$

$$\hookrightarrow \sim \int |\nabla u|^3 C$$

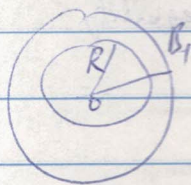
The term $|\nabla u|^2 u$ drops out.

harmless constant

$$\text{Then get } \int_{B_R} |\nabla u|^2 \leq \frac{C}{(1-R)^2} \left(\int |\nabla u|^3 \right) \leq \frac{C^2}{(1-R)^2}$$

$$n=5: H^1 \hookrightarrow \hookrightarrow L^{10/3-\delta} \quad \forall \delta > 0$$

Compact



$\Rightarrow (u|_{B_R})$ relatively compact in L^3

$\Rightarrow$ can pass through the limit in (*).

For $\varepsilon=0$: $\Delta v=0$

$$\int_{B_1} |\nabla v|^3 \leq 1, \quad |\nabla v| \leq C_R \text{ on } B_R$$

$$\frac{1}{\varepsilon^2} \int_{B_\varepsilon} |\nabla - (v)_{B_\varepsilon}|^3 dx \lesssim \frac{1}{\varepsilon^2} \varepsilon^3 \varepsilon^5 = \varepsilon^6$$

$$I(x, r) = \frac{1}{r^2} \int_{B_{x,r}} |v - (v)_{B_{x,r}}|^3 dx$$

$$J(x, r) = \frac{1}{r^5} \int_{B_{x,r}} |v - (v)_{B_{x,r}}|^3 dx$$

If $J(x, r) < \frac{\varepsilon}{r^3}$ (ε from the ε -regularity) then $J(x, r) \leq C\varepsilon^3 J(x, r)$
 $\exists C, \forall \varepsilon \in (0, \frac{1}{2}), \exists \varepsilon = \varepsilon(C) > 0$ such that $\nearrow$
 $\mathcal{C}^\alpha$ in some set $\mathcal{O}$

$$1) \quad \alpha < 1$$

$$2) \quad C\varepsilon^3 = \varepsilon^{3\alpha}$$

$$3) \quad J(x, r) < \frac{\varepsilon}{r^3} \quad (**)$$

$$\Rightarrow J(x, r) \leq \varepsilon^{3\alpha} J(x, r)$$

$$\text{Repeat: } J(x, r^k) \leq (\varepsilon^{3\alpha})^k J(x, r) \quad \forall k \in \mathbb{N}$$

$$\Rightarrow J(x, r) \leq \tilde{C} r^{3\alpha} J(x, r)$$

When $(**)$ is satisfied at x , it is also satisfied on a neighborhood of x .

$I(x, r) < \varepsilon \Rightarrow u$ smooth in a neighborhood of x .

If $\forall u \in L^2$ ($u \in H^1$) then by Poincaré inequality

$$\frac{1}{r} \int_{B_{x,r}} |u|^2 < \varepsilon_1 \Rightarrow I(x, r) < \varepsilon$$

$$f \in L^1, \quad S = \{x: \frac{1}{r} \int_{B_{x,r}} f \rightarrow ?\}, \quad \mathcal{H}^1(S) = 0.$$

Theorem: Suitable weak solution of $-(1-\alpha)\Delta u = |u|^2 u$.

$\exists S, \bar{S} = S, \mathcal{H}^1(S) = 0$ such that u is smooth in $\Omega \setminus S$.

NSE is similar, but much more technically difficult. See Caffarelli,
 Kohn, Nirenberg