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Math 8583: Theory of PDE

Homework #1

(1) Let Ω be an open subset of $\mathbb{R}^n$, $u \in C^2(\Omega)$ with $\Delta u(x) = 0$ in Ω. Let $A: \mathbb{R}^n \to \mathbb{R}^n$ be a linear orthogonal transformation. Note that as a consequence, A is invertible. Put $\Gamma = A^{-1}(\Omega)$, and define $v: \Gamma \to \mathbb{R}$, $v(y) = u(Ay)$. We'll show that $\Delta v(y) = 0$ on Γ.

Because A and A^{-1} are linear maps on finite-dimensional spaces, they are continuous. Thus A is also a homeomorphism. Thus $\Gamma = A^{-1}(\Omega)$ is an open subset of $\mathbb{R}^n$.

In the standard basis $(e_1, e_2, \dots, e_n)$ of $\mathbb{R}^n$, we write (and identify) each vector with its coordinates. Also, A is identified with its corresponding matrix.

$$x \equiv (x_1, \dots, x_n)^T, \quad y = (y_1, \dots, y_n)^T,$$

$$A = (a_{ij})_{1 \le i,j \le n}.$$

The orthogonality of A is equivalent to $AA^T = I_n$, i.e.

$$[AA^T]_{ik} = \delta_{ik} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

Thus, $\displaystyle\sum_{j=1}^n a_{ij} a_{kj} = \delta_{ik} \qquad (*)$

Let x and y in $\mathbb{R}^n$ be related by $x = Ay$. Then each x_i is a function of $y_1, y_2, \dots, y_n$. Specifically, $x_i = x_i(y_1, \dots, y_n) = \displaystyle\sum_{j=1}^n a_{ij} y_j$.

Then, $\dfrac{\partial x_i}{\partial y_j}(y_1, \dots, y_n) = a_{ij} \qquad (**)$

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By the definition of v, we have $v(y_1,\dots,y_n) = u(x_1,\dots,x_n)$. Then

$$\frac{\partial v}{\partial y_j}(y_1,\dots,y_n) = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(x_1,\dots,x_n)\,\frac{\partial x_i}{\partial y_j}(y_1,\dots,y_n)$$

$$\stackrel{(**)}{=} \sum_{i=1}^{n} a_{ij}\,\frac{\partial u}{\partial x_i}(x_1,\dots,x_n)$$

Taking the partial derivative with respect to y_j one more time, we get

$$\frac{\partial^2 v}{\partial y_j^2}(y_1,\dots,y_n) = \frac{\partial}{\partial y_j}\left(\sum_{i=1}^{n} a_{ij}\,\frac{\partial u}{\partial x_i}(x_1,\dots,x_n)\right)(y_1,\dots,y_n)$$

$$= \sum_{i=1}^{n} a_{ij}\,\frac{\partial}{\partial y_j}\left(\frac{\partial u}{\partial x_i}(x_1,\dots,x_n)\right)(y_1,\dots,y_n)$$

$$= \sum_{i=1}^{n} a_{ij} \sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left(\frac{\partial u}{\partial x_i}\right)(x_1,\dots,x_n)\,\frac{\partial x_k}{\partial y_j}(y_1,\dots,y_n)$$

$$\stackrel{(**)}{=} \sum_{i=1}^{n} a_{ij} \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k \partial x_i}(x_1,\dots,x_n)\, a_{kj}$$

$$= \sum_{i,k=1}^{n} a_{ij}\, a_{kj}\, \frac{\partial^2 u}{\partial x_k \partial x_i}(x_1,\dots,x_n).$$

Thus, $$\Delta v(y) = \sum_{j=1}^{n} \frac{\partial^2 v}{\partial y_j^2}(y_1,\dots,y_n) = \sum_{i,k=1}^{n}\left(\sum_{j=1}^{n} a_{ij}\,a_{kj}\right)\frac{\partial^2 u}{\partial x_k \partial x_i}(x_1,\dots,x_n)$$

$$\stackrel{(*)}{=} \sum_{i,k=1}^{n} \delta_{ik}\,\frac{\partial^2 u}{\partial x_k \partial x_i}(x_1,\dots,x_n)$$

$$= \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}(x_1,\dots,x_n)$$

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$$= \Delta u(x).$$

Since $\Delta u(x) = 0$ for all $x \in \Omega$, $\Delta v(y) = 0$ for all $y \in \Gamma$.

Very good! 10/10.

(2) Let $f \in C([-1,1])$. We will consider the problem of finding $u \in C^2([-1,1])$ such that

$$\begin{cases} u''(t) + f(t) = 0 & \forall t \in (-1,1), \\ u(-1) = u(1) = 0 \end{cases} \qquad (*)$$

Suppose that $u \in C^2([-1,1])$ is a solution of $(*)$. Then

$$u'(s) = u'(-1) + \int_{-1}^{s} u''(t)\,dt = u'(-1) - \int_{-1}^{s} f(t)\,dt \qquad \forall s \in (-1,1).$$

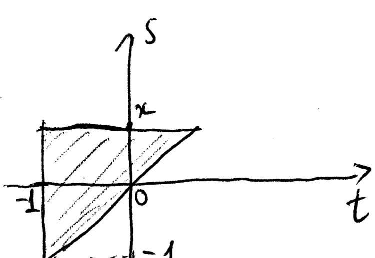
Then

$$\begin{aligned} u(x) &= u(-1) + \int_{-1}^{x} u'(s)\,ds \\ &= 0 + \int_{-1}^{x} \left[u'(-1) - \int_{-1}^{s} f(t)\,dt \right] ds \\ &= u'(-1)(x+1) - \int_{-1}^{x}\int_{-1}^{s} f(t)\,dt\,ds \qquad \forall x \in (-1,1) \end{aligned}$$

Because u is continuous at ± 1, we get

$$u(x) = u'(-1)(x+1) - \int_{-1}^{x}\int_{-1}^{s} f(t)\,dt\,ds \qquad \forall x \in [-1,1] \qquad (1)$$

Because $f \in C([-1,1])$, Fubini's theorem allows us to change the order of taking integration.



$$u(x) = u'(-1)(x+1) - \int_{-1}^{x}\int_{t}^{x} f(t)\,ds\,dt \qquad (2)$$

Thus, $u(x) = u'(-1)(x+1) - \int_{-1}^{x} (x-t) f(t)\,dt \qquad (**)$

Then $0 = u(1) = 2u'(-1) - \int_{-1}^{1} (1-t) f(t)\,dt.$

Thus $u'(-1) = \int_{-1}^{1} \frac{1-t}{2} f(t)\,dt$

Substituting this identity to (**), we obtain

$$u(x) = \int_{-1}^{1} \frac{1-t}{2}(x+1) f(t)\,dt - \int_{-1}^{x} (x-t) f(t)\,dt \qquad (3)$$

Then
$$u(x) = \int_{-1}^{1} \left[\frac{1-t}{2}(x+1) - (x-t)\, \chi_{[-1,x]}(t) \right] f(t)\,dt \qquad (4)$$

Put
$$G(x,y) = \frac{1-y}{2}(x+1) - (x-y)\,\chi_{[-1,x]}(y) \qquad \forall\, x,y \in [-1,1]$$

$$= \begin{cases} \underbrace{\frac{1-y}{2}(x+1) - (x-y)}_{\frac{(1-x)(1+y)}{2}} & \text{if } -1 \le y \le x \le 1, \\ \frac{1-y}{2}(x+1) & \text{if } -1 \le x < y \le 1. \end{cases}$$

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Then (4) gives us $\quad u(x) = \int_{-1}^{1} G(x,t) f(t)\,dt$.

Therefore, if the Problem (*) has a solution in $C^2([-1,1])$, it must be unique.

Conversely, we'll show that $\quad u(x) = \int_{-1}^{1} G(x,t) f(t)\,dt \quad$ is indeed a solution of Problem (*). Equation (4) holds true. Thus we have Equation (3). Then we have an identity similar to (**) with $u'(-1)$ being replaced by $\int_{-1}^{1} \frac{1-t}{2} f(t)\,dt$, namely

$$u(x) = (x+1) \int_{-1}^{1} \frac{1-t}{2} f(t)\,dt - \int_{-1}^{x} (x-t) f(t)\,dt. \qquad (5)$$

Thus $u \in C^1([-1,1])$ with $u(-1) = 0$ and

$$u(1) = 2 \int_{-1}^{1} \frac{1-t}{2} f(t)\,dt - \int_{-1}^{1} (1-t) f(t)\,dt = 0.$$

Because $\int_{-1}^{x}(x-t)f(t)\,dt = \int_{-1}^{x}\int_{t}^{x} f(t)\,ds\,dt \overset{\text{Fubini}}{=\!=} \int_{-1}^{x}\int_{-1}^{s} f(t)\,dt\,ds$, the Equation (5) is equivalent to

$$u(x) = (x+1)\int_{-1}^{1}\frac{1-t}{2}f(t)\,dt - \int_{-1}^{x}\int_{-1}^{s} f(t)\,dt\,ds \quad \forall x\in[-1,1] \quad (5')$$

Then $u\in C^1([-1,1])$ and
$$u'(x) = \int_{-1}^{1}\frac{1-t}{2}f(t)\,dt - \int_{-1}^{x} f(t)\,dt.$$

Then $u'\in C^1([-1,1])$ and
$$u''(x) = 0 - \frac{d}{dt}\int_{-1}^{x} f(t)\,dt = -f(x).$$

Thus, $u\in C^2([-1,1])$ and $u''(x)+f(x)=0$ for all $x\in(-1,1)$. Therefore, $u(x) = \int_{-1}^{1} G(x,t)f(t)\,dt$ is the unique solution in $C^2([-1,1])$ of the Problem (*).

(3) Let $f\in C([-1,1])$ and $u\in C^2([-1,1])$ be such that

$$\begin{cases} u''+f = 0 & \text{in } (-1,1),\\ u(-1)=u(1)=0. \end{cases}$$

Let $\gamma\in(0,1)$ be a constant. We'll show that there is a constant $N=N(\gamma)>0$ such that

$$\sup_{(-1,1)} (1-|x|)^{-\gamma}|u(x)| \le N \sup_{(-1,1)} (1-|y|)^{2-\gamma}|f(y)| \qquad (*)$$

By Problem 2, we have $u(x) = \int_{-1}^{1} G(x,y)\,dy$, where

$$G(x,y) = \begin{cases} \frac{1}{2}(1+y)(1-x) & \text{if } -1\le y\le x\le 1,\\ \frac{1}{2}(1-y)(1+x) & \text{if } -1\le x<y\le 1. \end{cases}$$

We notice that $0 \le G(x,y) \le \frac{1}{2}$ for all $x,y \in [-1,1]$. Put

$$A = \sup_{(-1,1)} (1-|y|)^{2-\gamma} |f(y)|$$

If $A = \infty$ then (*) is automatically satisfied for any choice of $N > 0$.

Consider the case $A < \infty$. We have

$$|u(x)| = \left| \int_{-1}^{1} G(x,y) f(y)\, dy \right| \le \int_{-1}^{1} G(x,y) |f(y)|\, dy$$

For $x \in (-1,1)$,

$$(1-|x|)^{-\gamma} |u(x)| \le (1-|x|)^{-\gamma} \int_{-1}^{1} G(x,y) |f(y)| dy$$

$$= (1-|x|)^{-\gamma} \int_{-1}^{1} \frac{G(x,y)}{(1-|y|)^{2-\gamma}} \left((1-|y|)^{2-\gamma} |f(y)| \right) dy$$

$$\le (1-|x|)^{-\gamma} \left(\int_{-1}^{1} \frac{G(x,y)}{(1-|y|)^{2-\gamma}} dy \right) A .$$

Thus, we need to find a constant $N = N(\gamma) > 0$ such that

$$h(x) := (1-|x|)^{-\gamma} \int_{-1}^{1} \frac{G(x,y)}{(1-|y|)^{2-\gamma}} dy \le N \qquad \forall x \in (-1,1). \qquad (**)$$

For $x \in (-1,1)$,

$$\int_{-1}^{1} \frac{G(x,y)}{(1-|y|)^{2-\gamma}} dy = (1-x) \underbrace{\int_{-1}^{x} \frac{1+y}{2(1-|y|)^{2-\gamma}} dy}_{\{1\}} +$$

$$+ (1+x) \underbrace{\int_{x}^{1} \frac{1-y}{2(1-|y|)^{2-\gamma}} dy}_{\{2\}} \qquad (1)$$

We will evalnate the terms {1} and {2} in two follawing cases.

- $0 \le x < 1$

If $-1 < y \le 0$ then $\dfrac{1+y}{2(1-|y|)^{2-\gamma}} = \dfrac{1+y}{2(1+y)^{2-\gamma}} = \dfrac{1}{2}(1+y)^{\gamma-1}$.

If $0 \le y \le x$ then $\dfrac{1+y}{2(1-|y|)^{2-\gamma}} = \dfrac{1+y}{2(1-y)^{2-\gamma}} \le \dfrac{1}{(1-y)^{2-\gamma}} = (1-y)^{\gamma-2}$.

Thus,

$$\{1\} = \int_{-1}^{0} \frac{1+y}{2(1-|y|)^{2-\gamma}}\,dy + \int_{0}^{x} \frac{1+y}{2(1-|y|)^{2-\gamma}}\,dy$$

$$\le \int_{-1}^{0} \frac{1}{2}(1+y)^{\gamma-1}\,dy + \int_{0}^{x}(1-y)^{\gamma-2}\,dy$$

$$= \frac{1}{2\gamma}(1+y)^{\gamma}\Big|_{-1}^{0} - \frac{(1-y)^{\gamma-1}}{\gamma-1}\Big|_{0}^{x}$$

$$= \frac{1}{2\gamma} - \frac{(1-x)^{\gamma-1}}{\gamma-1} + \frac{1}{\gamma-1}$$

$$\le \cancel{\frac{1-3\gamma}{2\gamma(1-\gamma)}}\ \frac{1}{2\gamma} + \frac{(1-x)^{\gamma-1}}{1-\gamma}. \qquad (2)$$

If $x \le y < 1$ then $\dfrac{1-y}{2(1-|y|)^{2-\gamma}} = \dfrac{1-y}{2(1-y)^{2-\gamma}} = \dfrac{1}{2}(1-y)^{\gamma-1}$.

Thus,

$$\{2\} = \int_{x}^{1} \frac{1-y}{2(1-|y|)^{2-\gamma}}\,dy \le \int_{x}^{1} \frac{1}{2}(1-y)^{\gamma-1}\,dy$$

$$= -\frac{1}{2\gamma}(1-y)^{\gamma}\Big|_{x}^{1}$$

$$= \frac{1}{2\gamma}(1-x)^{\gamma} \qquad (3)$$

Applying the estimates (2) and (3) into (1), we get

$$\int_{-1}^{1} \frac{G(x,y)}{(1-|y|)^{2-\gamma}}\,dy = (1-x)\{1\} + (1+x)\{2\}$$

$$\le (1-x)\left[\cancel{\frac{-1+3\gamma}{2\gamma(1-\gamma)}}\,\frac{1}{2\gamma} + \frac{(1-x)^{\gamma-1}}{1-\gamma}\right] + (1+x)\,\frac{1}{2\gamma}(1-x)^{\gamma}$$

$$= (1-x)\,\cancel{\frac{-1+3\gamma}{2\gamma(1-\gamma)}}\,\frac{1}{2\gamma} + \frac{(1-x)^{\gamma}}{1-\gamma} + \frac{1}{2\gamma}(1+x)(1-x)^{\gamma}.$$

Multiplying both sides by $(1-|x|)^{-\gamma} = (1-x)^{-\gamma}$, we get

$$h(x) \le (1-x)^{1-\gamma}\,\cancel{\frac{-1+3\gamma}{2\gamma(1-\gamma)}}\,\frac{1}{2\gamma} + \frac{1}{1-\gamma} + \frac{1}{2\gamma}(1+x)$$

$$\le \cancel{\frac{1+\gamma}{2\gamma(1-\gamma)}}\,\frac{1}{2\gamma} + \frac{1}{1-\gamma} + \frac{1}{\gamma}$$

$$= \frac{3-\gamma}{2\gamma(1-\gamma)} \qquad \forall x \in [0,1) \qquad (4)$$

$\underline{-1<x<0}$

If $-1<y<x$ then $\dfrac{1+y}{2(1-|y|)^{2-\gamma}} = \dfrac{1+y}{2(1+y)^{2-\gamma}} = \dfrac{1}{2}(1+y)^{\gamma-1}.$

Thus, $$\{1\} = \int_{-1}^{x} \frac{1}{2}(1+y)^{\gamma-1}\,dy = \frac{1}{2\gamma}(1+y)^{\gamma}\Big|_{-1}^{x} = \frac{(1+x)^{\gamma}}{2\gamma} \qquad (5)$$

If $x \le y < 0$ then $\dfrac{1-y}{2(1-|y|)^{2-\gamma}} = \dfrac{1-y}{2(1+y)^{2-\gamma}} \le \dfrac{1}{(1+y)^{2-\gamma}} = (1+y)^{\gamma-2}.$

If $0 \le y < 1$ then $\dfrac{1-y}{2(1-|y|)^{2-\gamma}} = \dfrac{1-y}{2(1-y)^{2-\gamma}} = \dfrac{1}{2}(1-y)^{\gamma-1}.$

Thus, $$\{2\} = \int_{x}^{0} \frac{1-y}{2(1-|y|)^{2-\gamma}}\,dy + \int_{0}^{1} \frac{1-y}{2(1-|y|)^{2-\gamma}}\,dy \le$$

$$\leq \int_x^0 \cancel{\tfrac{1}{2}}(1+y)^{\gamma-2}\,dy + \int_0^1 \frac{1}{2}(1-y)^{\gamma-1}\,dy$$

$$= \frac{(1+y)^{\gamma-1}}{\gamma-1}\Big|_x^0 - \frac{1}{2\gamma}(1-y)^{\gamma}\Big|_0^1$$

$$= \frac{1}{\gamma-1} - \frac{(1+x)^{\gamma-1}}{\gamma-1} + \frac{1}{2\gamma}$$

$$\leq \cancel{1+\gamma}\ \frac{1}{2\gamma} + \frac{(1+x)^{\gamma-1}}{1-\gamma} \qquad (6)$$

Applying the estimates (5) and (6) into (1), we get

$$\int_{-1}^{1} \frac{G(x,y)}{(1-|y|)^{2-\gamma}}\,dy = (1-x)\{1\} + (1+x)\{2\}$$

$$\leq \frac{1}{2\gamma}(1-x)(1+x)^{\gamma} + (1+x)\left[\frac{1}{2\gamma} + \frac{(1+x)^{\gamma-1}}{1-\gamma}\right]$$

$$= \frac{1}{2\gamma}(1-x)(1+x)^{\gamma} + \frac{1+x}{2\gamma} + \frac{(1+x)^{\gamma}}{1-\gamma}$$

Multiplying both sides by $(1-|x|)^{-\gamma} = (1+x)^{-\gamma}$, we get

$$h(x) \leq \frac{1}{2\gamma}(1-x) + \frac{(1+x)^{1-\gamma}}{2\gamma} + \frac{1}{1-\gamma}$$

$$\leq \frac{1}{\gamma} + \frac{1}{2\gamma} + \frac{1}{1-\gamma}$$

$$= \frac{3-\gamma}{2\gamma(1-\gamma)} \quad \cancel{\forall x \in (0 \quad (7)} \quad \forall x \in (-1,0) \qquad (7)$$

By (4) and (7), we have $h(x) \leq \frac{3-\gamma}{2\gamma(1-\gamma)}$ for all $x \in (-1,1)$. Therefore, we can choose $N = N(\gamma) = \frac{3-\gamma}{2\gamma(1-\gamma)}$ for each $\gamma \in (0,1)$.

Next, we'll show that the conclusion is false for $\gamma = 0$. Suppose otherwise. Then there is a constant $N > 0$ such that

$$\sup_{(-1,1)} |u(x)| \leq N \sup_{(-1,1)} (1-|y|)^2 |f(y)|$$

for all $f \in C([-1,1])$, where $u \in C^2([-1,1])$ satisfying

$$\begin{cases} u''(x) + f(x) = 0 & \text{in } (-1,1), \\ u(-1) = u(1) = 0 \end{cases} \qquad (I)$$

For each $\varepsilon > 0$, we put $\displaystyle f_\varepsilon(x) = \frac{1}{(1+\varepsilon-|x|)^2}$ for all $x \in [-1,1]$.

Then $f_\varepsilon \in C([-1,1])$. Let $u_\varepsilon \in C^2([-1,1])$ be the solution to Problem (I) with $f = f_\varepsilon$. We have

$$(1-|y|)^2 |f_\varepsilon(y)| = \frac{(1-|y|)^2}{(1+\varepsilon-|y|)^2} \leq 1 \quad \forall y \in (-1,1)$$

Thus, $\displaystyle\sup_{(-1,1)} (1-|y|)^2 |f_\varepsilon(y)| \leq 1$. Then $|u_\varepsilon(x)| \leq N$ for all $\varepsilon > 0$ and $x \in (-1,1)$.

In particular, $u_\varepsilon(0) \leq N \quad \forall \varepsilon > 0 \qquad (8)$

By the equation (5') in Problem 2, page 5, we have the formula

$$u_\varepsilon(x) = \underbrace{(x+1)\int_{-1}^{1} \frac{1-t}{2} f_\varepsilon(t)\,dt}_{A} - \underbrace{\int_{-1}^{x}\int_{-1}^{s} f_\varepsilon(t)\,dt\,ds}_{B} \qquad (9)$$

Because f_ε is an even function, we have

$$A = \frac{1}{2}\int_{-1}^{1} f_\varepsilon(t)\,dt - \frac{1}{2}\underbrace{\int_{-1}^{1} t f_\varepsilon(t)\,dt}_{0} = \int_{0}^{1} f_\varepsilon(t)\,dt.$$

Then $$A = \int_0^1 \frac{dt}{(1+\varepsilon-t)^2} = \frac{1}{1+\varepsilon-t}\Big|_0^1 = \frac{1}{\varepsilon} - \frac{1}{1+\varepsilon}.$$

For $-1 \leq x \leq 0$, we have

$$B = \int_{-1}^{x}\int_{-1}^{s} \frac{1}{(1+\varepsilon+t)^2}\,dt\,ds = \int_{-1}^{x} \frac{-1}{1+\varepsilon+t}\Big|_{-1}^{s}\,ds$$

$$= \int_{-1}^{x}\left(\frac{1}{\varepsilon} - \frac{1}{1+\varepsilon+s}\right)ds$$

Then $$B = \frac{x+1}{\varepsilon} - \log(1+\varepsilon+s)\Big|_{-1}^{x} = \frac{x+1}{\varepsilon} - \log(1+\varepsilon+x) + \log\varepsilon$$

Substituting A and B above into (9), we get

$$u_\varepsilon(x) = (x+1)\left(\frac{1}{\varepsilon} - \frac{1}{1+\varepsilon}\right) - \left(\frac{x+1}{\varepsilon} - \log(1+\varepsilon+x) + \log\varepsilon\right)$$

$\forall -1 < x \leq 0.$

Evaluating at $x=0$, we get

$$u_\varepsilon(0) = \frac{1}{\varepsilon} - \frac{1}{1+\varepsilon} - \left(\frac{1}{\varepsilon} - \log(1+\varepsilon) + \log\varepsilon\right)$$

$$= -\frac{1}{1+\varepsilon} + \log(1+\varepsilon) - \log\varepsilon$$

If $0<\varepsilon<1$ then $\quad u_\varepsilon(0) > -\frac{1}{2} + 0 - \log\varepsilon = \log\left(\frac{1}{\varepsilon}\right) - \frac{1}{2}.$

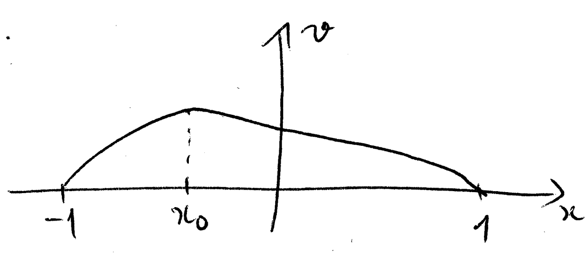
Thus $u_\varepsilon(0) \to \infty$ as $\varepsilon \to 0^+$. This contradicts with (8).

Excellent ! $\quad$ 10/10 -

④ (I excuse for using the symbol "v" instead of "U" to avoid possible confusion with the symbol "u" in my handwriting!)

Let K be a positive constant. We'll show that the problem of find $v \in C^2([-1,1])$ satisfying
$$\begin{cases} v'' + K|v'| + 1 = 0 & \text{in } (-1,1), \\ v(-1) = v(1) = 0 \end{cases} \qquad (*)$$
has a unique solution. Suppose that v is a solution in $C^2([-1,1])$ of Problem $(*)$. We'll find an explicit expression for v.

We have $v'' = -K|v'| - 1 < 0$. Thus, v' is strictly decreasing on $[-1,1]$. Since $v(-1) = v(1) = 0$, by Rolle's theorem, there exists $x_0 \in (-1,1)$ such that $v'(x_0) = 0$. Thus, $v'(x) > 0$ on $(-1, x_0)$ and $v'(x) < 0$ on $(x_0, 1)$. Thus, v is strictly increasing on $(-1, x_0)$ and strictly decreasing on $(x_0, 1)$. Thus v has a unique critical point (at x_0), which maximizes v because $v''(x_0) < 0$. Also, as a consequence,
$$v(x) > v(-1) = 0 \quad \forall -1 < x \le x_0,$$
$$v(x) > v(1) = 0 \quad \forall \ x_0 < x < 1.$$
Thus, $v(x) > 0$ for all $x \in (-1,1)$.



Problem (*) becomes
$$\begin{cases} v''+Kv'+1=0 & \text{in } (-1,x_0), \quad (1)\\ v''-Kv'+1=0 & \text{in } (x_0,1), \quad (2)\\ v(-1)=v(1)=0. \end{cases}$$

For $t\in(-1,x_0)$, we integrate (1) over $[-1,t]$:
$$0=\int_{-1}^{t}(v''+Kv'+1) = v'(t)-v'(-1)+Kv(t)-Kv(-1)+t-1$$
Thus $v'(t)+Kv(t)=-t+C_1$. Multiplying both sides by e^{Kt} we get
$$\frac{d}{dt}\left(ve^{Kt}\right) = -te^{Kt}+C_1e^{Kt}$$
For $x\in(-1,x_0)$, we integrate both sides over $[-1,x]$ using the fact that $v(-1)=0$:
$$v(x)e^{Kx} = -\int_{-1}^{x}te^{Kt}dt + C_1\int_{-1}^{x}e^{Kt}dt$$
Thus,
$$v(x) = -\frac{x+e^{-K}e^{-Kx}}{K}+\frac{1-e^{-K}e^{-Kx}}{K^2}+C_1\frac{1-e^{-K}e^{-Kx}}{K}$$
$$= -\frac{x}{K}+a_1+b_1e^{-Kx} \qquad \forall x\in(-1,x_0),$$
where a_1 and b_1 are contants.

For $t\in(x_0,1)$, we integrate (2) from 1 to t:
$$0=\int_{1}^{t}(v''-Kv'+1) = v'(t)-v'(1)-K\left(v(t)-v(1)\right)+t-1$$
Thus, $v'(t)-Kv(t)=-t+C_2$. Multiplying both sides by e^{-Kt} we get
$$\frac{d}{dt}\left(ve^{-Kt}\right) = -te^{-Kt}+C_2e^{-Kt}$$
For $x\in(x_0,1)$, we integrate both sides ~~ove~~ from 1 to x, using the fact that $v(1)=0$:
$$v(x)e^{-Kx} = -\int_{1}^{x}te^{-Kt}dt + C_2\int_{1}^{x}e^{-Kt}dt$$

Thus, $$v(x) = \frac{x - e^{-K}e^{Kx}}{K} + \frac{1-e^{-K}e^{Kx}}{K^2} - C_2\frac{1-e^{-K}e^{Kx}}{K}$$

$$= \frac{x}{K} + a_2 + b_2 e^{Kx} \qquad \forall x \in (x_0, 1).$$

We have proved that

$$v(x) = -\frac{x}{K} + a_1 + b_1 e^{-Kx} \qquad \forall -1 \leq x \leq x_0, \qquad (3)$$

$$v(x) = \frac{x}{K} + a_2 + b_2 e^{Kx} \qquad \forall x_0 \leq x \leq 1. \qquad (4)$$

where a_1, a_2, b_1, b_2 are constants. Consequently,

$$v'(x) = -\frac{1}{K} - Kb_1 e^{-Kx} \qquad \forall -1 < x < x_0, \qquad (5)$$

$$v'(x) = \frac{1}{K} + Kb_2 e^{Kx} \qquad \forall x_0 < x < 1. \qquad (6)$$

and

$$v''(x) = K^2 b_1 e^{-Kx} \qquad \forall -1 < x < x_0, \qquad (7)$$

$$v''(x) = K^2 b_2 e^{Kx} \qquad \forall x_0 < x < 1. \qquad (8)$$

Since v' is continuous, $v'(x_0^-) = v'(x_0^+) = v'(x_0) = 0$ (recall that x_0 is the maximizer of v). Then by (5) we get $e^{Kx_0} = -K^2 b_1$, or equivalently

$$b_1 = -\frac{e^{Kx_0}}{K^2} \qquad (9)$$

Similarly, by (6) with $x = x_0$ we have $\quad b_2 = -(1/K^2)e^{-Kx_0} \qquad (10)$

~~Since v is continuous, $v(x_0^-) = v(x_0^+)$. By (3) and~~ (4),

$$\cancel{-\frac{x_0}{K} + a_1 + b_1 e^{-Kx_0} = 0}, \qquad \cancel{\frac{x_0}{K} + a_2 + b_2 e^{+Kx_0} = 0}.$$

Since v is continuous, $v(x_0^-) = v(x_0^+)$. By (3) and (4),

$$-\frac{x_0}{K} + a_1 + b_1 e^{-Kx_0} = \frac{x_0}{K} + a_2 + b_2 e^{Kx_0}.$$

Replacing b_1 and b_2 at (9) and (10), we get

$$-\frac{x_0}{K} + a_1 - \frac{1}{K^2} = \frac{x_0}{K} + a_2 - \frac{1}{K^2}$$

Thus, $$x_0 = \frac{K}{2}(a_1 - a_2) \qquad (11)$$

We have the boundary conditions:

$$0 = v(-1) \overset{(3)}{=\!=} \frac{1}{K} + a_1 + b_1 e^K,$$

$$0 = v(1) \overset{(4)}{=\!=} \frac{1}{K} + a_2 + b_2 e^K.$$

Thus, $$a_1 = -b_1 e^K - \frac{1}{K} \overset{(9)}{=\!=} \frac{e^{Kx_0} e^K}{K^2} - \frac{1}{K} \qquad (12)$$

$$a_2 = -b_2 e^K - \frac{1}{K} \overset{(10)}{=\!=} \frac{e^{-Kx_0} e^K}{K^2} - \frac{1}{K} \qquad (13)$$

With a_1 and a_2 given at (12) and (13), we rewrite (11) as follows.

$$x_0 = \frac{K}{2} e^K \left(\frac{e^{Kx_0}}{K^2} - \frac{1}{K^2 e^{Kx_0}} \right) = \frac{e^K}{2K}(e^{Kx_0} - e^{-Kx_0})$$

Thus, x_0 is a zero in $(-1,1)$ of the function

$$f(t) = \frac{e^K}{2K}(e^{Kt} - e^{-Kt}) - t$$

We see that $f(0) = 0$, and $$f'(t) = \frac{e^K}{2}(e^{Kt} + e^{-Kt}) - 1$$

$$> \frac{e^{Kt} + e^{-Kt}}{2} - 1$$

$$\geq 0$$

Thus f is strictly increasing on $(-1,1)$ and hence $t=0$ is the only zero of f on $(-1,1)$. Therefore, $x_0 = 0$. By (9) and (10), $b_1 = b_2 = -\frac{1}{K^2}$.
By (12) and (13), $a_1 = a_2 = \frac{e^K}{K^2} - \frac{1}{K}$. Now we can rewrite (3) and (4) as follows.

$$v(x) = -\frac{x}{K} + \frac{e^K}{K^2} - \frac{1}{K} - \frac{e^{-Kx}}{K^2} \qquad \forall -1 \le x \le 0, \tag{14}$$

$$v(x) = \frac{x}{K} + \frac{e^K}{K^2} - \frac{1}{K} - \frac{e^{Kx}}{K^2} \qquad \forall\, 0 \le x \le 1. \tag{15}$$

In a more compact form,

$$v(x) = \frac{|x|}{K} - \frac{e^{K|x|}}{K^2} + \frac{e^K}{K^2} - \frac{1}{K} \tag{16}$$

Now we'll double check that v given by (14), (15) is indeed a solution to Problem (1), (2). It is easy to see that $v(-1) \overset{(14)}{=} 0$ and $v(1) \overset{(15)}{=} 0$.
Because $v(0^-) = v(0^+) = \frac{e^K}{K^2} - \frac{1}{K^2} - \frac{1}{K}$, v is continuous. By (14) and (15),

$$v'(x) = \begin{cases} -\frac{1}{K} + \frac{e^{-Kx}}{K} & -1 < x < 0, \\ \frac{1}{K} - \frac{e^{Kx}}{K} & 0 < x < 1. \end{cases}$$

Because $v'(0^-) = v'(0^+) = 0$, $v \in C^1([-1,1])$. We have

$$v''(x) = \begin{cases} -e^{-Kx}, & -1 < x < 0, \\ -e^{Kx}, & 0 < x < 1. \end{cases}$$

Because $v''(0^-) = v''(0^+) = -1$, $v \in C^2([-1,1])$. By substituting the above formulae into (1) and (2), we see that v really satisfies the equations.

Therefore, the function v given at (16) is the unique solution to Problem (*).

Let $p \in C([-1,1])$ and $u \in C^2([-1,1])$ be such that $|p(x)| \leq K$ in $[-1,1]$ and

$$\begin{cases} u'' + pu' + 1 = 0 & \text{in } (-1,1), \\ u(-1) = u(1) = 0 \end{cases} \qquad (**)$$

We'll show that $0 \leq u \leq v$ in $(-1,1)$ where v is given by Equation (16).

Denote $L\phi := \phi'' + p\phi'$. Then

$$\begin{cases} Lu = u'' + pu' = -1 < L(0) & \text{in } (-1,1), \\ u(-1) = 0 \geq 0, \ u(1) = 0 \geq 0 \end{cases}$$

By the Comparison Principle for ODE of second order, $u(t) \geq 0$ for all $t \in (-1,1)$. It now remains to show that $u(t) \leq v(t)$ for all $t \in (-1,1)$.

Suppose for a moment that we have $u(0) \leq v(0)$.

For $t \in (-1,0)$, v satisfies Equation (1) (page 13), i.e. $v'' + Kv + 1 = 0$. Moreover, as discussed from the beginning of the problem, $v'(t) \geq 0$. Thus,

$$Lv(t) = v'' + p(t)v' = \underbrace{(v'' + Kv')}_{-1} + \underbrace{(p(t) - K)}_{\leq 0} \underbrace{v'}_{\geq 0} \leq -1 = Lu(t)$$

Because $Lv(t) \leq Lu(t)$ and $v(-1) = 0 \geq u(-1)$, $v(0) \geq u(0)$, by the Comparison Principle applied on $(-1,0)$, we have $v(t) \geq u(t)$ for all $t \in (-1,0)$.

Similarly, for $t \in (0,1)$, v satisfies Equation (2) (page 13), i.e. $v'' - Kv' + 1 = 0$. Moreover, as discussed from the beginning of the problem, $v'(t) \leq 0$. Then

$$Lv(t) = v'' + p(t)v' = \underbrace{(v'' - Kv')}_{-1} + \underbrace{(K + p(t))}_{\geq 0}\underbrace{v'}_{\leq 0} \leq -1 = Lu(t).$$

Because $Lv(t) \leq Lu(t)$ and $v(0) \geq u(0)$, $v(1) = 0 \geq u(1)$, by the Comparison Principle applied on $(0,1)$, we have $v(t) \geq u(t)$ for all $t \in (0,1)$.

Therefore, the whole problem reduces to showing that $u(0) \leq v(0)$.

By Equation (16), $\quad v(0) = \dfrac{e^K - K - 1}{K^2}$.

Because $u(-1) = u(1) = 0$, by Rolle's theorem there exists $x_1 \in (-1,1)$ such that $u'(x_1) = 0$. Put $\quad q(x) = \int_0^x p(t)\,dt \quad \forall x \in [-1,1]$.

Because u solves Problem (**) we have

$$u''(t) + p(t)u'(t) + 1 = 0 \qquad \forall t \in (-1,1).$$

Multiplying both sides by $e^{q(t)}$, we get $\dfrac{d}{dt}\left(e^{q(t)}u'(t)\right) + e^{q(t)} = 0$.

For $x \in (-1,1)$, taking the integral both sides from x_1 to x, we get

$$e^{q(x)}u'(x) + \int_{x_1}^{x} e^{q(t)}\,dt = 0$$

Hence,

$$u'(x) = -\int_{x_1}^{x} e^{q(t)-q(x)}\,dt \qquad \forall x \in (-1,1) \qquad (17)$$

By the definition of function q, we have

$$q(t) - q(x) = \int_x^t p(s)\,ds, \quad \forall x, t \in [-1,1].$$

Because $|p(s)| \leq K$, we have

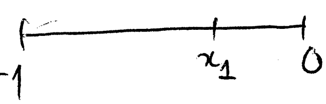
$$|q(t) - q(x)| \leq K|t-x| \quad \forall x, t \in [-1,1]. \qquad (18)$$

We consider two following cases.

- $\underline{x_1 \leq 0}$

By Equation (17), $u'(x) = \int_x^{x_1} e^{q(t)-q(x)}\,dt \quad \forall x \in (-1,1)$.

For $x \in (-1,0)$, we have $u'(x) \leq \int_x^0 e^{q(t)-q(x)}\,dt$



$$\leq \int_x^0 e^{K(t-x)}\,dt \quad (\text{by } (18))$$

$$= \frac{e^{K(t-x)}}{K}\Bigg|_{t=x}^{t=0}$$

$$= \frac{e^{-Kx}-1}{K}.$$

Thus, $u'(x) \leq \dfrac{e^{-Kx}-1}{K}$ for all $x \in (-1,0)$. Taking the integral over $x \in [-1,0]$, we get

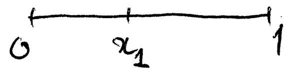
$$u(0) = \int_{-1}^0 u'(x)\,dx \leq \int_{-1}^0 \frac{e^{-Kx}-1}{K}\,dx$$

$$= \left(-\frac{1}{K^2}e^{-Kx} - \frac{x}{K}\right)\Bigg|_{-1}^0$$

$v(0)$ $$= -\frac{1}{K^2} + \frac{e^K}{K^2} - \frac{1}{K} =$$

Therefore, $u(0) \leq v(0)$.

$\underline{x_1 > 0}$ By Equation (17), $-u'(x) = \int_{x_1}^{x} e^{q(t)-q(x)}\,dt$.



For $x \in (0,1)$, $-u'(x) \leq \int_0^x e^{q(t)-q(x)}\,dt$

$$\leq \int_0^x e^{K(x-t)}\,dt \quad (\text{by } (18))$$

Thus, $-u'(x) \leq \left.\dfrac{e^{K(x-t)}}{-K}\right|_{t=0}^{t=x} = \dfrac{e^{Kx}-1}{K}$ for all $x \in (0,1)$.

Taking the integral over $x \in [0,1]$, we get

$$u(0) = \int_0^1 -u'(x)\,dx \leq \int_0^1 \frac{e^{Kx}-1}{K}\,dx = \left.\left(\frac{e^{Kx}}{K^2} - \frac{x}{K}\right)\right|_0^1$$

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$$= \frac{e^K}{K^2} - \frac{1}{K^2} - \frac{1}{K} = v(0).$$

Therefore, $u(0) \leq v(0)$.

⑤ Let $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a bounded smooth solution of the problem

$$\begin{cases} u_t = u_{xx}, & \forall t>0,\ \forall x \in \mathbb{R} \\ u(0,x) = \exp(-x^4), & \forall x \in \mathbb{R} \end{cases}$$

We'll show that $u(t,0)$ is not analytic at $t=0$. Suppose otherwise.

Then the function $v(t) = u(t,0)$ is given by a power series in a neighborhood $[0,r)$ of 0 in $[0,\infty)$.

$$v(t) = v(0) + \sum_{j=1}^{\infty} \frac{v^{(j)}(0)}{j!} t^j \qquad \forall\, 0 \leq t < r.$$

Consequently, the radius of convergence of this series must be $\geq r$.

$$\sum = 10 + 10 + 10 + 10 + 10 = 50/50.$$

Thus, $\limsup_{j\to\infty} \sqrt[j]{\left|\frac{v^{(j)}(0)}{j!}\right|} \leq \frac{1}{r} < \infty.$ (1)

We'll prove by induction that $v^{(j)}(t) = D_x^{2j} u(t,0).$ (2).

For $j=1$, $v'(t) = D_t u(t,0) = D_x^2 u(t,0)$. Hence (2) is true. Suppose that (2) is true for j. Then

$$v^{(j+1)}(t) = \frac{d}{dt}\left(v^{(j)}(t)\right) = D_t\left(D_x^{2j} u\right)(t,0) = D_x^{2j}(D_t u)(t,0)$$
$$= D_x^{2j}(D_x^2)(t,0) = D_x^{2(j+1)}(t,0).$$

Thus, (2) is true for $j+1$. Therefore, it is true for all $j \geq 1$.

As a consequence, $v^{(j)}(0) = D_x^{2j} u(0,0) \quad \forall j \geq 1$ (3)

Because $u(0,x) = \exp(-x^4)$, $u(0,x)$ is analytic at $x=0$ and

$$u(0,x) = 1 + \sum_{m=1}^{\infty} \frac{(-x^4)^m}{m!} = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} x^{4m} \quad (4)$$

On the other hand, we know that

$$u(0,x) = u(0,0) + \sum_{j=1}^{\infty} \frac{D_x^j u(0,0)}{j!} x^j \quad (5)$$

Equating the coefficients of x^{4m} in (4) and (5), we get

$$\frac{(-1)^m}{m!} = \frac{D_x^{4m} u(0,0)}{(4m)!}$$

Then $D_x^{4m} u(0,0) = (-1)^m \frac{(4m)!}{m!}$ (6)

In (3), with $j=2m$, we have $v^{(2m)}(0) = D_x^{4m} u(0,0) \stackrel{(6)}{=} \frac{(-1)^m (4m)!}{m!}.$

Thus, $\left|\frac{v^{(2m)}(0)}{(2m)!}\right| = \frac{(4m)!}{(2m)!\, m!} = \frac{(4m)!}{(2m)!(2m)!} \frac{(2m)!}{m!\, m!} m!$

$$= C_{4m}^{2m} C_{2m}^{m} m! > m!$$

Hence, $\sqrt[2m]{\left|\frac{v^{(2m)}(0)}{(2m)!}\right|} > \sqrt[2m]{m!} = \left(\sqrt[m]{m!}\right)^{1/2} \longrightarrow \infty$ as $m \to \infty$.

Thus, $\lim\limits_{m\to\infty} \sqrt[2m]{\left|\frac{v^{(2m)}(0)}{(2m)!}\right|} = \infty.$

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This contradicts with (1).

For such expressions the ratio test is simpler in application.