Worksheet

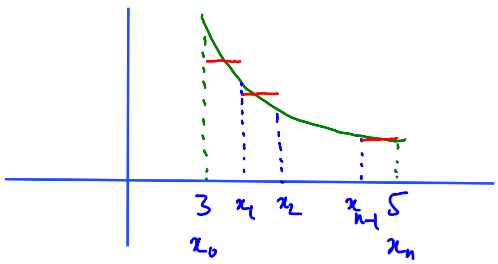
03/11/2020

Name: ______________________________

1. Let us compute approximately the integral $I = \int_3^5 \frac{1}{x^2-1}dx$ by midpoint rule (call the sum M_n) with equally spaced sample points $3 = x_0 < \ldots < x_n = 5$.

   (a) Write M_n using sigma notation.

   (b) Find n such that M_n approximates I with error not exceeding $\epsilon = 10^{-4}$.



$$h = \frac{5-3}{n} = \frac{2}{n}.$$

$$x_0 = 3, \ x_1 = 3+h, \ x_2 = 3+2h, \ldots$$

$$x_k = 3 + kh = 3 + \frac{2k}{n}.$$

The midpoint of the interval $[x_k, x_{k+1}]$ is

$$x_k^* = \frac{x_k + x_{k+1}}{2} = \frac{1}{2}\left(3 + \frac{2k}{n} + 3 + \frac{2(k+1)}{n}\right)$$

$$= 3 + \frac{2k+1}{n}.$$

The midpoint Riemann sum is

$$M_n = hf(x_0^*) + hf(x_1^*) + \cdots + hf(x_{n-1}^*)$$

$$= \sum_{k=0}^{n-1} h\, f(x_k^*) = \sum_{k=0}^{n-1} h\, f\left(3 + \frac{2k+1}{n}\right)$$

$$M_n = \frac{2}{n}\sum_{k=0}^{n-1} \frac{1}{\left(3 + \frac{2k+1}{n}\right)^2 - 1}.$$

We have

$$|M_n - I| \le \underbrace{\frac{(b-a)^3}{24n^2}}_{\frac{(5-3)^3}{24n^2} = \frac{1}{3n^2}} \max|f''| = \frac{1}{3n^2}\max_{[3,5]}|f''| \qquad (*)$$

We have

$$f(x) = \frac{1}{x^2-1}$$

$$f'(x) = \frac{-2x}{(x^2-1)^2}$$

$$f''(x) = \frac{-2(x^2-1)^2 - (-2x)2(2x)(x^2-1)}{(x^2-1)^4}$$

$$= \frac{-2(x^2-1) + 8x^2}{(x^2-1)^4}$$

$$= \frac{6x^2+2}{(x^2-1)^4}$$

For $x \in [3,5]$,

$$|f''(x)| = \frac{6x^2+2}{(x^2-1)^4} \leq \frac{6(5)^2+2}{(3^2-1)^4} = \frac{19}{512}.$$

Therefore, $\displaystyle\max_{[3,5]} |f''| \leq \frac{19}{512}$

From $(*)$, we have

$$|M_n - I| \leq \frac{1}{3n^2} \times \frac{19}{512}.$$

To make sure that $|M_n - I| < 10^{-4}$, we only need to choose n such that

$$\frac{1}{3n^2} \times \frac{19}{512} < 10^{-4}.$$

Any $n \geq 12$ would do it.

2. Approximate the integral in Problem 1 using Simpson's rule with $n = 4$. How large should n be such that the Simpson sum S_n approximates I with error not exceeding $\epsilon = 10^{-4}$?

We will skip Simpson's rule.