Worksheet

2/7/2020

Name: ________________________________

1. Consider the function $f(x) = \frac{\sin x}{x}$. Find a polynomial P such that

$$\max_{x\in[1,2]} |f(x) - P(x)| < 10^{-3}.$$

We have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \underbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}}_{q_{2n+1}(x)} + \underbrace{\cdots\cdots}_{r_{2n+1}(x)}$$

Divide both sides by x:

$$\underbrace{\frac{\sin x}{x}}_{f(x)} = \underbrace{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots + (-1)^n \frac{x^{2n}}{(2n+1)!}}_{\frac{q_{2n+1}(x)}{x}} + \underbrace{\cdots\cdots}_{\frac{r_{2n+1}(x)}{x}}$$

We see that $\frac{q_{2n+1}(x)}{x}$ must be the $(2n)$'th Taylor polynomial of $f(x)$,

and $\frac{r_{2n+1}(x)}{x}$ must be the corresponding error term.

Thus

$$P_{2n}(x) = \frac{q_{2n+1}(x)}{x} = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k+1)!},$$

$$R_{2n}(x) = \frac{r_{2n+1}(x)}{x}.$$

We have

$$f(x) = P_{2n}(x) + R_{2n}(x)$$

We choose $P(x) = P_{2n}(x)$, with n to be determined

How to determine n?

$$f(x) - P(x) = f(x) - p_{2n}(x) = R_{2n}(x).$$

We need to choose n such that

$$\max_{x\in[1,2]} |R_{2n}(x)| < 10^{-3}.$$

It is difficult to apply Lagrange's theorem directly to function f. But it is easier to apply Lagrange's theorem to function $g(x) = \cos x$. The error term of f and the error term of g are related to each other by

$$R_{2n}(x) = \frac{r_{2n+1}(x)}{x}.$$

By Lagrange's theorem,

$$r_{2n+1}(x) = \frac{g^{(2n+2)}(c)}{(2n+2)!} x^{2n+2}$$

for some c in between 0 and x. Then

$$R_{2n}(x) = \frac{r_{2n+1}(x)}{x} = \frac{g^{(2n+2)}(c)}{(2n+2)!} x^{2n+1}.$$

Take the absolute value of both sides:

$$|R_{2n}(x)| = \frac{|g^{(2n+2)}(c)|}{(2n+2)!} |x|^{2n+1} \qquad (*)$$

Because $g^{(2n+2)}$ can only be cos, sin, −cos, −sin, its values are always in between -1 and 1. Thus, $|g^{(2n+2)}(c)| \le 1$ (regardless of where c is).

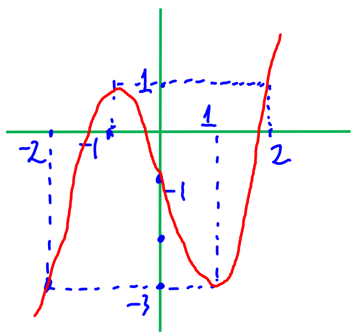
Also, because $x \in [1,2]$, $|x| \le 2$. By $(*)$ we get

$$|R_{2n}(x)| \le \frac{1}{(2n+2)!} 2^{2n+1} \qquad \forall x \in [1,2].$$

To guarantee that $\max_{x\in[1,2]} |R_{2n}(x)| \le 10^{-3}$, we only need to find a large n such that $\frac{1}{(2n+2)!} 2^{2n+1} < 10^{-3}$. By testing with calculator, one can choose $n = \cdots$

2. Use a suitable numerical method to find an approximation of each root of the polynomial $x^3 - 3x - 1$ with allowed error $\epsilon = 10^{-2}$.

Sketch the graph:



$f(x) = x^3 - 3x - 1$

The polynomial has 3 roots because

$$\left.\begin{array}{l} f(-2) = -3 < 0 \\ f(-1) = 1 > 0 \end{array}\right\} \text{root } r_1 \in (-2,-1)$$

$$\left.\begin{array}{l} f(-1) = 1 > 0 \\ f(0) = -1 < 0 \end{array}\right\} \text{root } r_2 \in (-1,0)$$

$$\left.\begin{array}{l} f(1) = -3 < 0 \\ f(2) = 1 > 0 \end{array}\right\} \text{root } r_3 = (1,2)$$

Note that f has only 3 roots because it is a polynomial of degree 3.

We need to find approximate values of r_1, r_2, r_3 with allowed error 10^{-2}.

We should use Bisection method because it tells us how many steps to take to get an approximate value with the prescribed error.

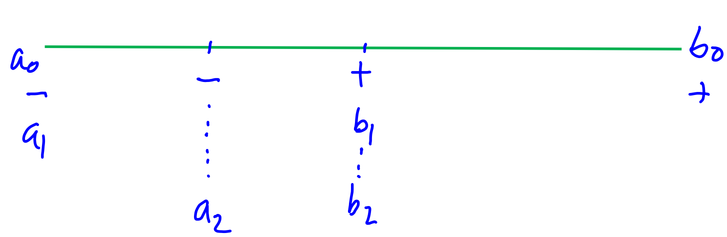
- Compute r_1

The initial interval is $[a_0, b_0] = [-2,-1]$.

The number of steps we need to take is

$$n \geq \log_2\left(\frac{b_0 - a_0}{\varepsilon}\right) - 1 = \log_2\left(\frac{1}{10^{-2}}\right) - 1 \approx 5.6$$

Thus, 6 steps would be enough.



$c_0 = -1.5$

$f(c_0) = f(-1.5) > 0$

$c_1 = -1.75$

$f(c_1) = f(-1.75)$

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