Worksheet

2/5/2020

Name: ______________________________________

1. Consider a sequence defined recursively as follows.

$$x_{n+1} = \frac{1}{4}(x_n^3 - 3x_n + 6), \qquad x_0 = 1.5.$$

(a) Use your pocket calculator to guess the limit of this sequence. Then use the recursive formula to verify that this number is truly a limit of (x_n).

(b) Find the order of convergence. If the order of convergence is 1, find the linear rate of convergence.

$$x_1 = \frac{1}{4}(x_0^3 - 3x_0 + 6) = 1.21875$$

$$x_2 = \frac{1}{4}(x_1^3 - 3x_1 + 6) \approx 1.0386$$

$$x_3 = \frac{1}{4}(x_2^3 - 3x_2 + 6) \approx 1.0011$$

$$x_4 = \frac{1}{4}(x_3^3 - 3x_3 + 6) \approx 1.00000096$$

The limit of (x_n) seems to be 1.

To check that this limit is indeed 1, we put $a = \lim x_n$.

Then take the limit of both sides of the equation

$$x_{n+1} = \frac{1}{4}(x_n^3 - 3x_n + 6).$$

We get

$$a = \frac{1}{4}(a^3 - 3a + 6)$$

which is equivalent to $a^3 - 7a + 6 = 0$,

which is equivalent to $(a-1)(a^2 + a - 6) = 0$, which is equivalent to

$$(a-1)(a-2)(a+3)=0$$

This equation gives $a=1$ or $a=2$ or $a=-3$.

Because the sequence is never close to 2 or -3, the limit must be $a=1$.

To find the order of convergence, we subtract 1 from both sides of the iteration formula:

$$\begin{aligned} x_{n+1}-1 &= \frac{1}{4}(x_n^3-3x_n+6)-1 = \frac{1}{4}(x_n^3-3x_n+2) \\ &= \frac{1}{4}(x_n-1)(x_n^2+x_n-2) \\ &= \frac{1}{4}(x_n-1)(x_n-1)(x_n+2) \\ &= \frac{1}{4}(x_n-1)^2(x_n+2). \end{aligned}$$

Now take the absolute value of both sides:

$$|x_{n+1}-1| = \frac{1}{4}|x_n-1|^2|x_n+2|$$

In other words,

$$e_{n+1} = \frac{1}{4}e_n^2|x_n+2| \approx \frac{1}{4}e_n^2|1+2| = \frac{3}{4}e_n^2 \quad \text{(when } n \text{ is large)}$$

Thus, the order of convergence is $p=2$.

2. We will use Newton's method to find approximate solutions of the system

$$\begin{cases} x + \sin y &= 1, \\ xy + \sin x &= 1. \end{cases}$$

(a) Write an iteration formula of the Newton's method.

(b) Do 3 iterations with $(x_0, y_0) = (1, 1)$.

We'll work on this after the midterm exam.